Preliminary Examination in Partial Differential Equations
January 6, 2011

Instructions This is a three-hour examination. The exam is divided into two parts. You need to solve a total of five problems. You must do at least two problems from each part. Please indicate clearly on your test papers which five problems are to be graded.

You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

Notation. Euclidean n space is denoted by $\mathbf{R}^n$ while $B(x,r) = \{y \in \mathbf{R}^n : |y-x| < r\}$.

PART I

(1) Let h be a solution to Laplace's equation in $B(0,1)$. That is, h has two continuous partials in $B(0,1)$ and $\Delta h = 0$ in $B(0,1)$.
   (a) State and prove the mean value property for h in $B(0,1)$.
   (b) Use (a) to show that if also $h \geq 0$ in $B(0,1)$, then there is a constant c depending only on n such that $h(x) \leq ch(y)$ whenever $x, y \in B(0, 1/4)$.

(2) Let Ω be a bounded domain with smooth boundary in $\mathbf{R}^n$. Let $\Omega_T = \Omega \times (0,T)$ and let $\bar{\Omega}_T$ be the closure of Ω_T. Show that there is at most one real valued function u in $C^2(\bar{\Omega}_T)$ which satisfies

$$\begin{cases} u_t - \Delta u = -2u^3 & \text{in } \Omega \\ u(x,0) = 0, & x \in \Omega \\ u(x,t) = 0, & x \in \partial\Omega,\ 0 < t < T. \end{cases}$$

(3) A real-valued continuous function u is said to be a weak solution to the wave equation in an open set $U \subset \mathbf{R}^2$ if

$$\int_U (\phi_{xx} - \phi_{tt})\, u \, dx dt = 0$$

whenever $\phi \in C_0^\infty(U)$.
   (a) Show that if u is a classical solution to the wave equation (i.e. $u \in C^2(U)$ and $u_{xx} = u_{tt}$ pointwise) in U, then u is a weak solution to the wave equation in U.
   (b) Show that if $u(x,t) = f(x-t)$ where f is a continuous function on $\mathbf{R}^2$, then u is a weak solution to the wave equation in $\mathbf{R}^2$.

(4) Consider the first-order equation

$$xu_y - yu_x = u$$

with initial condition $u(x,0) = h(x)$ where h is a given smooth function.

(a) Write down the characteristic differential equations for this first order PDE.

(b) Using the method of charactersitics solve the PDE in (a). Express your answer in terms of the function h.

(c) What can you say about uniqueness of your solution?

PART II

(5) Let $u(x) = \log|x|$ when $x \in B(0,1) \subset \mathbf{R}^n$.
Prove that $u \in W^{1,p}(B(0,1))$ when $1 \leq p < n$.

(6) Prove or disprove:

(a) $W_0^{1,2}(\mathbf{R}^n)$ is dense in $W^{1,2}(\mathbf{R}^n)$.

(b) $W_0^{1,2}(B(0,1))$ is dense in $W^{1,2}(B(0,1))$.

You may assume inequalities of Sobolev or Poincaré provided you first state these inequalities.

(7) $v \in W^{1,2}(B(0,1))$ is a weak solution to

$$Lv = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial v}{\partial x_i}(x)\right) = 0 \text{ in } B(0,1),$$

provided

$$\int_{B(0,1)} \sum_{i,j=1}^{n} a_{ij}(x) v_{x_i} \zeta_{x_j}\, dx = 0$$

whenever $\zeta \in C_0^\infty(B(0,1))$. In this display $(a_{ij}(x))$ is a symmetric matrix with measurable coefficients, satisfying

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, x \in \bar{B}(0,1),$$

for some $0 < \lambda, \Lambda < \infty$, and all $\xi \in \mathbf{R}^n$. Show that if v is a weak solution to $Lv = 0$ in $B(0,1)$ then there exists $c \geq 1$, depending only on n, λ, Λ, such that

$$\int_{B(0,1/2)} |\nabla v|^2 dx \leq c \int_{B(0,1)} v^2 dx\,.$$

(8) Let v, a_{ij} be as in (7) and suppose also that $|v| \leq M < \infty$ in $B(0,1)$. Let $\psi \in C^2(\mathbf{R})$ be a convex function. Prove that $w = \psi \circ v$ is a weak subsolution to L in the sense that if $\zeta \geq 0 \in C_0^\infty(B(0,1))$, then

$$\int_{B(0,1)} \sum_{i,j=1}^{n} a_{ij}(x) w_{x_i} \zeta_{x_j} \, dx \leq 0.$$

You may assume that $w \in W^{1,2}(B(0,1))$.