Preliminary Examination in Partial Differential Equations

4 January 2013

Instructions

This is a three-hour examination. The exam is divided into two parts. You should attempt at least two questions from each part and a total of five questions. Please indicate clearly on your test paper which five questions are to be graded.

Provide complete solutions to each problem and give as much detail as possible. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly the theorems and definitions you are using.

PART I

Throughout the exam, $B(x,r)$ denotes the open ball in $\mathbf{R}^n$ with center x and radius r. We use D_s to denote the partial derivative with respect to a variable s.

1. Let u be a C^2 function in $B(0,1)$ and let $\phi(r) = \frac{1}{\omega_{n-1}r^{n-1}} \int_{\partial B(0,r)} u(y)\, d\sigma(y)$. Here ω_{n-1} denotes the $n-1$-dimensional measure of the boundary of the unit ball in $\mathbf{R}^n$.

   (a) Show that
   $$\phi'(r) = \frac{1}{\omega_{n-1}r^{n-1}} \int_{B(0,r)} \Delta u(y) dy,$$
   where Δ is the Laplacian.

   (b) A C^2 function is said to be sub-harmonic if $\Delta u \geq 0$. If u is subharmonic in the unit ball $B(0,1)$, show that we have
   $$u(0) \leq \frac{1}{\omega_{n-1}r^{n-1}} \int_{\partial B(0,r)} u(y)\, dy.$$

2. Let $u(x,t)$ be a solution of the heat equation, $D_t u - \Delta u = 0$ for (x,t) in $U = B(0,1) \times (0,1)$ and assume that u is in $C^2(\bar{U})$. Let $\Gamma = (B(0,1) \times \{0\}) \cup (\partial B(0,1) \times [0,1])$.

   Show that
   $$\max_{\Gamma} u = \max_{\bar{U}} u.$$

3. Suppose that u is a C^2 solution of the equation
   $$2D_{x_1} D_{x_2} u - D^2_{x_2} u = 0$$
   in $\mathbf{R}^2$. Show that there are functions F and G so that
   $$u(x_1, x_2) = F(x_1) + G(x_1 + 2x_2).$$

4. Find the solution of the Cauchy problem in $\mathbf{R}^2$.
   $$\begin{cases} 2D_{x_1} u + D_{x_2} u = 1 \\ u(x_1, x_1) = x_1^2. \end{cases}$$

PART II

5. Let $f(x) = 1/|x|$ for x in the unit ball $B(0,1) \subset \mathbb{R}^3$.

   (a) Find $g_i = \partial f/\partial x_i$, the ordinary derivative of f when $x \neq 0$.

   (b) Since g_i is defined except at zero, g_i is defined almost everywhere on $B(0,1)$. Show that the weak derivative of f with respect to x_i, $D_{x_i} f$, is the function g_i.

   (c) For which p is f in the Sobolev space $W^{1,p}(B(0,1))$?

6. (a) Give the definition of the Sobolev spaces: $W^{1,2}(B(0,1))$, and $W_0^{1,2}(B(0,1))$.

   (b) Let $u \in W^{1,2}(B(0,1))$, and suppose that $\limsup_{x \to y} u(x) \leq 0$ whenever $y \in \partial B(0,1)$. Show that $u^+ = \max(u,0) \in W_0^{1,2}(B(0,1))$.

7. Recall that a function f on $B(0,1)$ is said to be Lipschitz with constant M provided
$|f(x) - f(y)| \leq M\,|x - y|$ for all $x, y \in B(0,1)$ and some $M < \infty$.

   (a) Use weak compactness of $L^2(B(0,1))$ and difference quotients to show that $f \in W^{1,2}(B(0,1))$.

   (b) Let $n = 1$ (so that $B(0,1)$ is the open interval $(-1,1)$) and let $f, f_m, m = 1, 2, \ldots$, be Lipschitz functions with constant M on $B(0,1)$. Suppose $\lim_{m\to\infty} f_m(x) = f(x)$ uniformly on $B(0,1)$. Prove or disprove by example: $f_m'(x) \to f'(x)$ as $m \to \infty$ almost everywhere with respect to Lebesgue measure on $B(0,1)$.

8. Consider the Neumann problem for Laplace's equation in $B(0,1) \subset \mathbb{R}^n$, $n \geq 2$:

$$\begin{cases} \Delta u(x) = f(x), & x \in B(0,1), \\ \nu \cdot \nabla u(x) = 0, & x \in \partial B(0,1). \end{cases}$$

Here, ν denotes the outer unit normal to $\partial B(0,1)$. Let H be the space of functions on $B(0,1)$ with $\nabla u = (D_{x_1} u, \ldots, D_{x_n} u) \in L^2(B(0,1))$ and $\int_{B(0,1)} u\,dx = 0$.

   (a) Show that H is a Hilbert space with the inner product

$$(u,v) = \int_{B(0,1)} \nabla u \cdot \nabla v\,dx.$$

Hint: If $\|u\|$ is the norm associated to this inner product and $\|\cdot\|_1 = \left(\int_{B(0,1)} |u|^2 + |\nabla u|^2\,dx\right)^{1/2}$, show that for some constants c and C, we have

$$c\|u\|_1 \leq \|u\| \leq C\|u\|_1.$$

(b) Give a definition of a weak solution to the Neumann problem.

(c) If f in $L^2(B(0,1))$ and $\int_{B(0,1)} f\,dx = 0$, show that we have a unique weak solution to the Neumann problem in the space H.

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