Preliminary Examination
Partial Differential Equations
June 2003

Instructions

This is a three-hour examination. You need to solve a total of **five problems**. The exam is divided into two parts. **You must do at least two problems from each part.**

Please indicate clearly on your test papers which five problems are to be graded.

You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

PART ONE

(1) Given that a function is harmonic in an open set $U \subset \mathbb{R}^n$ if, and only if, it satisfies the mean value property, show that the limit of a sequence of harmonic functions converging uniformly on compact subsets of U is harmonic in U.

(2) A function u is said to be <u>weakly</u> harmonic in $\mathbb{R}^n$ if u is continuous in $\mathbb{R}^n$ and $\int u\,\Delta\phi\,dx = 0$ whenever ϕ is infinitely differentiable on $\mathbb{R}^n$ with compact support. Show that weakly harmonic functions are harmonic. That is, show u is harmonic in $\mathbb{R}^n$.

(3) Let $D = \{(x, y) \,:\, x^2 + y^2 < 1\}$.
(a) Suppose $u \in C(\bar{D}) \cap C^2(D)$ and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u^3 \text{ in } D.$$

Show that u cannot obtain a positive maximum at a point of D.

(b) Show that the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = u^3 \text{ in } D \\ \\ u = 0 \text{ on } \partial D = \{(x,y) : x^2 + y^2 = 1\} \end{cases}$$

has no solution other than $u \equiv 0$.

(c) Let $S = \{(x,y) \in \mathbb{R}^2 : |x| < 1,\ |y| < 1\}$. Suppose $u \in C(\bar{S}) \cap C^2(S)$ and

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = -1 \text{ in } S \\ \\ u = 0 \text{ on } \partial S. \end{cases}$$

Find a bound for $u(0,0)$.

(4) Given that functions of the form $a|x|^{2-n} + b$, are harmonic in $\mathbb{R}^n \setminus \{0\}$, where a, b, are constants, $n > 2$, $x = (x_1, \ldots, x_n)$ and $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$. Let $u > 0$ be harmonic in $\{x : |x| < 1\}$ and continuous in $\{x : |x| \leq 1\}$. Suppose for some x_0 with $|x_0| = 1$ that $u(x_0) = 0$. Prove the *Hopf boundary maximum principle* :

$$\lim_{r \to 1^-} \frac{u(rx_0)}{1-r} > 0.$$

PART TWO

(5) Suppose Ω is a bounded, open, connected subset of $\mathbb{R}^n$. Suppose $u \in L^1(\Omega)$ and

$$\int_\Omega u\varphi_{x_i}\, dx = 0$$

for each $\varphi \in C_c^1(\Omega)$ and $1 \leq i \leq n$.

Show that there is a constant $c \in \mathbb{R}$ such that $u = c$ a.e. in Ω,

(6) Let $Lu = \Delta u + \sum_{i=1}^n b_i(x) D_i u + c(x) u$. Prove that if $c < 0$ is bounded in a bounded open set $U \subset \mathbb{R}^n$ and $u \in C^2(U) \cap C^0(\overline{U})$ satisfies $Lu = f$ in U then

$$\sup_U |u| \leq \sup_{\partial U} |u| + \sup_U \left|\frac{f}{c}\right|.$$

(7) Let

$$L = -\sum_{ij=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})$$

where the $a_{ij}(x)$ are bounded and measurable, $a_{ij} = a_{ji}$, and L is uniformly elliptic. Assume that $u \in H^1(U)$ is a bounded weak solution to

$$Lu = 0, \text{ in } U$$

Let $\phi \in C^\infty(R)$ be convex. Set $w = \phi(u)$. Show that $w \in H^1(U)$ and w is a weak subsolution, i.e.

$$B[w, \psi] \leq 0, \text{ for each } \psi \in H_0^1(U), \text{ with } \psi \geq 0$$

where $B[\cdot, \cdot]$ is the bilinear form associated with L.

(8) Suppose $U \subset \mathbb{R}^n$ is a bounded, smooth domain. Let the operator L be as in the previous question. Use the difference quotient method to prove: if $f \in L^2(U)$ and $u \in H^1(U)$ is a weak solution of $Lu = f$ in U then, in fact, $u \in H^2_{\text{loc}}(U)$ and for any open subset $V \subset\subset U$

$$\|D^2 u\|_{L^2(V)} \leq C(\|f\|_{L^2(U)} + \|u\|_{L^2(U)})$$