Preliminary Examination in Partial Differential Equations

June 6, 20~~11~~ 12

Instructions This is a three-hour examination.The exam is divided into two parts. You need to solve a total of five problems. You must do at least two problems from each part. Please indicate clearly on your test papers which five problems are to be graded.

You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

Notation. Euclidean n space is denoted by $\mathbf{R}^n$ while $B(x,r) = \{y \in \mathbf{R}^n : |y-x| < r\}$.

PART I

(1) Let $\Omega \subset \mathbf{R}^n$ be open and connected. Prove that for a harmonic function u on Ω, we have

$$|\nabla u(x_0)| \leq \frac{C}{r^{n+1}} \int_{B(x_0,r)} |u(y)|\, dy,$$

for any ball $B(x_0,r) \subset\subset \Omega$, where $C > 0$ is a constant depending only on the dimension n.

(2) (a) Suppose $g \in L^\infty(\mathbf{R}) \cap C(\mathbf{R})$. Prove that the function $u(x,t)$, for $x \in \mathbf{R}$ and $t > 0$ given by

$$u(x,t) = \frac{1}{(4\pi t)^{1/2}} \int_{\mathbf{R}} e^{-\frac{|x-y|^2}{4t}} g(y)\, dy, t > 0,$$

solves the heat equation.

(b) Prove that $\lim_{(x,t)\to(x_0,0);t>0} u(x,t) = g(x_0)$ for each $x_0 \in \mathbf{R}$.

(c) Prove that:

$$\left|\frac{\partial u}{\partial x}(x,t)\right| \leq \frac{C_0}{t^{1/2}} \|g\|_\infty,$$

where the constant $C_0 > 0$ is independent of g and $t > 0$.

(3) Let $U = \{(x,t) \; : \; x \in \mathbf{R}^3, t > 0\}$ and suppose that $u \in C^2(U) \cap C^1(\overline{U})$ is a solution to the nonhomogeneous wave equation:

$$\begin{cases} u_{tt} - \Delta u = f \text{ in } U \\ u(x,0) = g(x) \text{ for } x \in \mathbf{R}^3 \\ u_t(x,0) = h(x) \text{ for } x \in \mathbf{R}^3 \end{cases}$$

. The function $f \in C_0^\infty(U)$ and the initial conditions $g, h \in C_0^\infty(\mathbf{R}^3)$. Assuming appropriate decay of the function $u(x,t)$ on time slices in a neighborhood of infinity, use the energy method to derive the estimate:

$$E(t)^{1/2} \leq E(0)^{1/2} + t^{1/2} \left(\int_{\mathbf{R}^3 \times [0,t]} |f(x,s)|^2 \, dx \, ds \right)^{1/2},$$

where $E(t)$ is the energy given by

$$E(t) = \int_{\mathbf{R}^3} (|Du|^2 + |u_t|^2) \, dx,$$

with

$$E(0) = \int_{\mathbf{R}^3} (|Dg|^2 + |h|^2) \, dx.$$

PART II

(4) Let $\Omega \subset \mathbf{R}^n$ be a bounded, open, connected set. Let $f \in L^2(\Omega)$ and $\overrightarrow{g} = (g_1, \ldots, g_n)$ be a vector-valued function with $g_i \in L^2(\Omega)$, for $i = 1, 2, \ldots, n$.

(a) Prove that the boundary value problem

$$\begin{cases} -\Delta u = f - \nabla \cdot \overrightarrow{g} \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

has a unique weak solution in $H_0^1(\Omega)$.

(b) show that the weak solution satisfies

$$\|u\|_{H^1(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|\overrightarrow{g}\|_{L^2(\Omega)}),$$

for a finite constant $C > 0$ depending only on d and Ω

(5) (a) State the Rellich-Kondrachov Compactness Theorem for a bounded C^1 domain $\Omega \subset \mathbf{R}^n$.

(b) Apply part (a) to prove the Poincaré inequality for $W^{1,2}(\Omega)$: There is a finite constant $C > 0$, depending only on Ω, so that for all $v \in W^{1,2}(\Omega)$,

$$\int_\Omega (v - v_\Omega)^2 \, dx \le C \int_\Omega |Dv|^2 \, dx,$$

where v_Ω is the average of v over Ω.

(6) Let us denote the coordinates of $x \in \mathbf{R}^n$ as $x = (x', x_n) \in \mathbf{R}^{n-1} \times \mathbf{R}$. Let $1 \le p < \infty$. If $u \in W^{1,p}(U) \cap C(\mathbf{R}^{n-1} \times \{0\})$, we define the trace map $T : W^{1,p}(U) \to L^p(\mathbf{R}^{n-1} \times \{0\})$ by $(Tu)(x') = u(x', 0)$. Prove that T extends to a bounded linear map $\tilde{T} : W^{1,p}(U) \to L^p(\mathbf{R}^{n-1} \times \{0\})$ and that there is a constant $C_p > 0$, depending only on p, so that for all $u \in W^{1,p}(\mathbf{R}^n)$, we have

$$\|\tilde{T}u\|^p_{L^p(\mathbf{R}^{n-1})} \le C_p \int_{\mathbf{R}^n} [|u(x)|^p + |Du(x)|^p] \, dx.$$

where, if $u \in W^{1,p}(U) \cap C(\mathbf{R}^{n-1} \times \{0\})$, we have

$$\|\tilde{T}u\|^p_{L^p(\mathbf{R}^{n-1})} = \int_{\mathbf{R}^{n-1}} |u(x', 0)|^p \, dx'.$$