Preliminary Examination in Partial Differential Equations

June 2021

Instructions

This is a three-hour examination. You are to work a total of **five problems**. The exam is divided into two parts. **You must do at least two problems from each part**.

Please indicate clearly on your test papers which five problems are to be graded. You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy bits from two different problems. Indicate clearly what theorems and definitions you are using.

PART ONE

Problem 1. *Let* $u \in C(\bar{B}(0,1))$ *be given by the Poisson integral on the unit ball in* $\mathbb{R}^d$

$$u(x) = \frac{1-|x|^2}{\omega_{d-1}} \int_{\partial B(0,1)} \frac{u(y')}{|x-y'|^d}\, d\sigma(y'), \qquad x \in B(0,1).$$

Here ω_{d-1} *is the surface area of the unit ball and* $d\sigma$ *is surface measure. Prove that if* $u \geq 0$ *in* $\bar{B}(0,1)$*, then*

$$\sup_{B(0,1/2)} u \leq C \inf_{B(0,1/2)} u$$

where the constant C *depends only on the dimension. Your proof should give an explicit value for* C*.*

Problem 2. *Suppose that* $u(x,t)$ *is smooth and solves the heat equation*

$$u_t - \Delta u = 0$$

in $\mathbb{R}^n \times (0,\infty)$*.*

(1) *Show that the function* $u_\lambda(x,t) = u(\lambda x, \lambda^2 t)$ *also solves the heat equation for each* $\lambda \in \mathbb{R}$*.*
(2) *Using your result from part 1, show that the function*

$$v(x,t) = x \cdot Du(x,t) + 2t\, u_t(x,t)$$

also solves the heat equation.

Problem 3. *Let* u *be a smooth solution of the wave equation* $u_{tt} - \Delta u = 0$ *in* $\mathbb{R}^{d+1}$ *and suppose that*

$$u(x,0) = u_t(x,0) = 0, \qquad |x| \leq 1.$$

Prove that $u = 0$ *in the set* $\{(x,t) : |x| \leq 1-t,\ 0 < t < 1\}$*.*

Problem 4. *Suppose* U *is a smooth bounded domain in* $\mathbb{R}^2$*, and* $b \in C^\infty(U)$*. Suppose moreover that* $u \in C^1(U) \cap C(\bar{U})$ *solves the equation*

$$u_x + b u_y + u_y^2 = 0$$

in U*, with* $u > 0$ *on* ∂U*. Show that* $u > 0$ *in* U*.*

PART TWO

Problem 5. (1) *Let u lie in a Sobolev space $W^{1,p}(\mathbb{R}^d)$ and $v \in C_0^\infty(\mathbb{R}^d)$, the space of functions which are smooth and compactly supported in $\mathbb{R}^d$. Prove uv is weakly differentiable and prove that*

$$\frac{\partial}{\partial x_j}(uv) = u\frac{\partial}{\partial x_j}v + v\frac{\partial}{\partial x_j}u.$$

(2) *Let $d \geq 3$. If u and v lie in the Sobolev space $W^{1,2}(\mathbb{R}^d)$, prove that uv lies in $W^{1,d/(d-1)}(\mathbb{R}^d)$.*

Problem 6. *Suppose $U = (a,b) \subset \mathbb{R}$, and $u \in W_0^{1,1}(U)$. Show that there is a continuous function $\tilde{u}$ such that $u = \tilde{u}$ almost everywhere in U.*

Hint: first, show that

$$\sup_{x \in U} |f(x)| \leq \|f\|_{W_0^{1,1}(U)}$$

for all f in a dense subset of $W_0^{1,1}(U)$.

Problem 7. *Suppose that U is a bounded domain with smooth boundary, that $f \in L^2(U)$, and consider the* biharmonic problem

$$\Delta^2 u = f \quad \text{in } U$$

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial U.$$

(1) *Say what it mean for a function $u \in H_0^2(U)$ to be a weak solution to the biharmonic problem.*

(2) *Prove that there exists a unique weak solution of the biharmonic problem for given $f \in L^2(U)$.*

Problem 8. *Suppose U is a smooth bounded domain in $\mathbb{R}^n$ and $f \in L^2(U)$.*

(1) *Show that for $h > 0$, the equation*

$$-\Delta u + hu = f \text{ in } U$$

$$u|_{\partial U} = 0$$

has a unique weak solution $u \in H_0^1(U)$.

(2) *Suppose for each $h > 0$, u_h is the unique weak solution to the problem*

$$-\Delta u_h + hu_h = f \text{ in } U$$

$$u_h|_{\partial U} = 0$$

Show that $u = \lim_{h \to 0^+} u_h$ exists in the H^1 sense, and u is a weak solution to $-\Delta u = f$ with $u = 0$ on ∂U.