# Preliminary Examination in Partial Differential Equations

June 2026

## Instructions

This is a three-hour exam. You are to work a total of **five problems**.

This exam is divided into two parts. **You must do at least two problems from each part**. Please indicate clearly on your exam paper which five problems are to be graded.

You should provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than to solutions of the easy parts from two different problems.

Please indicate clearly what theorems and definitions you are using.

# Part I

In the following we assume that the domain $U \subset \mathbb{R}^n$ is bounded, smooth, and connected.

1. This problem concerns harmonic functions.

   (a) State the Mean Value Property for harmonic functions in a domain $U$.

   (b) Suppose that $u \in C^2(U) \cap C(\overline{U})$ and that $\Delta u = 0$ in $U$. Show that
   $$\max_{\overline{U}} u = \max_{\partial U} u.$$

2. Suppose that $u = u(x,t)$ is smooth and solves the heat equation
   $$u_t - \Delta u = 0$$
   in $\mathbb{R}^n \times (0,\infty)$.

   (a) Show that $u_\lambda(x,t) = u(\lambda x, \lambda^2 t)$ also solves the heat equation.

   (b) Use part (a) to show that
   $$v(x,t) = x \cdot (Du)(x,t) + 2tu_t(x,t)$$
   also solves the heat equation.

   *Hint*: Find $\partial u_\lambda / \partial \lambda$.

3. Let $c > 0$ be a constant. Assume that $u$ is a classical solution of the equation
   $$\begin{cases} u_t - \Delta u + cu = 0, \quad (x,t) \in U \times (0,\infty), \\ u = 0 \quad \text{on } \partial U, \\ u(x,0) = g(x), \end{cases}$$
   where $g \in C(\overline{U})$ is nonnegative. Show that there is a constant $C$ that depends only on $g$ so that
   $$u(t,x) \le Ce^{ct}.$$

   *Hint*: Let $v(t,x) = e^{-ct}u(x,t)$. What equation does $v$ satisfy?

4. Assume that $u \in C^2(\mathbb{R}^d \times [0,\infty);\mathbb{R})$ solves the equation

$$\begin{cases} u_{tt} - \Delta u + u^3 = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ u = u_t = 0 & \text{on } \mathbb{R}^d \times \{t = 0\} \end{cases}$$

Assume also that, for each fixed $t \geq 0$, the function $g(x) := u(x,t)$ is compactly supported. Show that $u \equiv 0$.

*Hint*: Consider

$$E[u](T) = \int_{\mathbb{R}^d} \frac{1}{2}(|u_t(x,T)|^2 + |\nabla_x u(x,T)|^2) + \frac{1}{4}u(x,T)^4\,dx$$

# Part II

1. i) Give the definition of the Sobolev space $H^1(U)$.

   ii) Assume that $u \in H^1(U)$, and let $v = \sin u$. Compute the weak derivatives of $v$, and show that $v \in H^1(U)$.

2. Consider the equation

$$\begin{cases} -\Delta u + e^{|x|^2} u = f \quad \text{in } U \\ u = 0 \text{ on } \partial U \end{cases}$$

   i) What does it mean for $u \in H_0^1(U)$ to be a weak solution?

   ii) Show that has a unique weak solution $u \in H_0^1(U)$ for all $f \in L^2(U)$.

3. Suppose $u \in H^1(B(0,1))$ is a weak solution to the elliptic equation

$$\sum_{i,j=1}^{n} \partial_{x_j}(a_{ij}(x) u_{x_i}) = 0$$

   on $B(0,1)$, with $a_{ij}$ smooth on $\overline{B(0,1)}$ and $a_{ij} = a_{ji}$.

   (a) Show there exists a constant $C$ depending only on $A$, such that

$$\int_{B(0,1)} |\eta|^2 |\nabla u|^2 dx \leq C \int_{B(0,1)} |\nabla \eta|^2 |u|^2 dx$$

   for each $\eta \in C_c^1(U)$.

   (b) Show that for each $0 < r < 1$, there exists $C$ depending only on $r$ and $A$ such that

$$\int_{B(0,r)} |\nabla u|^2 dx \leq \frac{C}{(R-r)^2} \int_{B(0,1)} |u|^2.$$

4. Consider

$$Lu = -\sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} + c(x) u$$

where the coefficients $(a_{ij}), c \in C(\bar{U})$, satisfy $a_{ij} = a_{ji}$, $c(x) \geq \mu > 0$, and the ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \qquad \text{for a.e. } x \in \mathbb{R}^n \text{ and for any } \xi \in \mathbb{R}^n, \tag{1}$$

for a fixed $\theta > 0$.

Let $f \in L^\infty(U)$. Show that any classical solution $u \in C^2(U) \cap C(\bar{U})$ to $Lu = f$ satisfies

$$\max_{\bar{U}} u \leq \max_{\partial U} u$$

Hint: Use the weak maximum principle for $u - v$ for a suitably chosen $v$.