DEPARTMENT OF MATHEMATICS

Topology Preliminary Examination
January 13, 2014

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1 Consider the topology on $\mathbf{R}$ given by the subbasis consisting of open rays (a, ∞).

(a) Given a subset $A \subset \mathbf{R}$, describe the closure $\bar{A}$ in this topology.
(b) Consider the sequence $x_n = n$. Does it converge? If so, to what?

2 Let $D \subset X$ be a dense subset of a metric space X. Suppose $f : X \longrightarrow Y$ restricts to a homeomorphism $f|_D : D \cong Y$. Show that $D = X$.

3 Let G be a topological group. Prove that every two components of G are homeomorphic.

4 Let X be a locally compact second countable space. Prove that there is a sequence $K_1 \subset K_2 \subset K_3 \subset \cdots$ of compact subspaces of X such that X is the union of the interiors of K_n's:

$$X = \bigcup_n \operatorname{Int} K_n.$$

5 Let $\mathrm{GL}(n, \mathbf{R})$ denote the space of invertible $n \times n$ matrices, and let $\mathrm{SL}(n, \mathbf{R})$ denote the space of $n \times n$ matrices of determinant 1. Consider the map

$$\phi : \mathrm{GL}(n, \mathbf{R}) \longrightarrow \mathrm{SL}(n, \mathbf{R})$$

that divides the first column of M by $\det(M)$.

(a) Is ϕ continuous for all $n \geq 1$? Why or why not?
(b) Is ϕ a retraction for all $n \geq 1$? Why or why not?
(c) Is ϕ a covering map for all $n \geq 1$? Why or why not?

6 Let P^n denote n-dimensional real projective space, i.e. the space obtained by identifying x and $-x$ for all $x \in S^n$. Prove that P^{3k} is not homeomorphic to $\prod_{i=1}^k S^1 \times \prod_{j=1}^k S^2$.

7 Let E be connected and locally connected, and let $p : E \longrightarrow B$ be a covering map. Suppose $f : S^2 \longrightarrow E$ is continuous and that $p \circ f$ is nullhomotopic. Show that f must be nullhomotopic.

8 Let X be a compact surface with a cell decomposition which has a single 0-cell, three 1-cells a, b, and c, and a single 2-cell attached according to $abacb^{-1}c^{-1}$. Determine the homeomorphism type of the surface.