DEPARTMENT OF MATHEMATICS

Topology Preliminary Examination
January 8, 2016

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1 Show that the intersection of two compact subspaces A and B of a Hausdorff space X is compact.

2 Show that a retract of a Hausdorff space must be a closed subset.

3 Recall that if $X_1, X_2, \ldots, X_n$ are disjoint topological spaces, the *disjoint union* $X_1 \sqcup X_2 \sqcup \ldots \sqcup X_n$ is the union $X = X_1 \cup X_2 \cup \ldots \cup X_n$ with the topology in which $U \subset X$ is open if and only if $U \cap X_i$ is open for all $i = 1, 2, \ldots, n$.

Show that if X has finitely many connected components, then X is homeomorphic to the disjoint union of its components.

4 For two spaces X, Y, let $\mathrm{Map}(X, Y)$ be the set of continuous maps $X \longrightarrow Y$ with the compact-open topology. Let $Y \subset Z$ and let $i : Y \longrightarrow Z$ be the inclusion map. Show that the induced map

$$i_\star : \mathrm{Map}(X, Y) \longrightarrow \mathrm{Map}(X, Z),$$

defined by $i_\star(f) = i \circ f$, is a homeomorphism onto its image.

5 Let $f : S^1 \longrightarrow T^2$ be the inclusion $f(x) = (x, 1)$ and $g : S^1 \longrightarrow T^2$ be the inclusion $g(x) = (1, x)$. Show that f is **not** homotopic to g.

6 Let G be a topological group with multiplication μ and identity e. (Also assume G is path connected and locally path connected.) If $(\tilde{G}, p)$ is a connected cover of G and $\tilde{e} \in \tilde{G}$ satisfies $p(\tilde{e}) = e$, show that there is a unique multiplication on $\tilde{G}$ for which $\tilde{e}$ is the identity and p is a homomorphism.

7 Show that there is no covering map, in either direction, between the projective plane $\mathbf{RP}^2$ and the Klein bottle K.

8 Suppose A is a retract of X with inclusion i and retraction r. If $i_\star\pi_1(A)$ is a normal subgroup of $\pi_1(X)$, show that

$$\pi_1(X) \cong i_\star\pi_1(A) \times \mathrm{Ker}(r_\star).$$