Topology Preliminary Exam
Jan. 2022

1. Let $X \subset \mathbb{R}^2$ be the subspace consisting of points (x, y) such that at least one of x and y is rational. Show that X is connected.

2. The subspace $\mathbb{Z} \subset \mathbb{R}$ is also a subgroup (under addition). We can therefore consider both the quotient $\mathbb{R}/\mathbb{Z}$, in which the subspace $\mathbb{Z}$ is collapsed to a point, and also the quotient denoted $\mathbb{Z}\backslash\mathbb{R}$, in which we are passing to orbits under the $\mathbb{Z}$-action on $\mathbb{R}$. Show that these quotient spaces are **not** homeomorphic.

3. Let X be a topological space. Let $O(X)$ be the set of open subsets of X. For $\{K_i\}_{i \in I}$, a set of compact subsets of X, let

$$V_{\{K_i\}} = \{U \subset X | U \text{ is open and } \exists i \in I \text{ such that } K_i \subset U\}.$$

(a) Show that the subsets of $O(X)$ of the form $V_{\{K_i\}}$ define a topology on $O(X)$. Hint: The case in which some K_i is empty and the case in which the indexing set I is empty are both important.

(b) Produce a homeomorphism $O(X) \cong \mathrm{Map}(X, S)$, where $S = \{0, 1\}$ with topology $\{\emptyset, \{1\}, \{0, 1\}\}$ and Map denotes the mapping space from X to S with the compact–open topology.

4. Show that there does **not** exist a retraction $r \colon X \longrightarrow A$ in the following situations:

(a) $X = S^1 \times D^2$ and A is the boundary torus $A = S^1 \times S^1$.

(b) $X = S^2 \subset \mathbb{R}^3$ and

$$A = \{(x, y, 0) \mid x^2 + y^2 = 1\} \cup \{(0, y, z) \mid y^2 + z^2 = 1\}.$$

5. Let M be a compact orientable surface of genus 2. Prove there exists

$$f : M \to S^1$$

continuous, which does not lift to a continuous map from M to $\mathbb{R}$.
(Here 'lift' refers to the exponential covering map $\mathbb{R} \to S^1$.)

6. Let X be the space obtained from the torus $S^1 \times S^1$ by attaching a Moebius band M by a homeomorphism from the boundary circle of M to $S^1 \times \{x_0\} \subset S^1 \times S^1$ (for $x_0 \in S^1$ fixed). Calculate $\pi_1(X, x_0)$.

7. Let X be a path–connected space with $H_1(X, \mathbb{Z}) = 0$.

(a) What can you deduce about $\pi_1(X, x_0)$?

(b) Describe how you would construct a space X satisfying $\pi_1(X, x_0) \neq H_1(X) = 0$. (Recall that the commutator subgroup is normal.)

8. Let A, B be path connected open (nonempty) subsets of S^n so that $A \cup B = S^n$.

(a) If $n \geq 2$, prove $A \cap B$ is path connected.

(b) Is the conclusion still true for $n = 1$?