Topology Preliminary Exam
January, 2026

On grading: A necessary condition to pass this exam is to completely solve one point-set question and one of the more algebraic questions. An excellent exam will completely solve five problems and have some partial credit on the remaining questions.

1. (*a*) Let X be a Hausdorff space such that all subsets of X are compact. Prove that X is finite.

(*b*) Give, with proof, an example of an infinite topological space X such that all subsets of X are compact.

2. Let $\{0,1\}$ be the space with two elements and the discrete topology. Let

$$X = \{0,1\} \times \{0,1\} \times \{0,1\} \times \dots,$$

the countably infinite product of $\{0,1\}$'s. Recall that a space is separable if it has a countable dense subset.

(*a*) Show that if X is given the product topology, then it is separable.

(*b*) Show that if X is given the box topology, then it is not separable.

3. Let G be a topological group.

(*a*) Suppose that G is locally path-connected. Write G_e for the path-component of the identity element. Show that G is homeomorphic to $G_e \times \pi_0(G)$, where $\pi_0(G)$ is given the discrete topology.

(*b*) Show that (a) fails if G is not locally path-connected.

4. Let Z denote the quotient of the square $[0,1] \times [0,1]$ obtained by identifying the four corners $(0,0) \sim (1,0) \sim (0,1) \sim (1,1)$. Prove that Z is not simply connected.

5. Let M and N be connected n-manifolds, and let $M\#N$ be their connected sum. Show that, if $n \geq 3$, then $\pi_1(M\#N)$ is the free product $\pi_1(M) * \pi_1(N)$.

6. Let X be path-connected, locally path connected, and semilocally simply-connected. Assume that $p\colon X \to X$ is a non-injective covering map. Prove that $\pi_1(X, x_0)$ is infinite for any choice of base point $x_0 \in X$.

7. Suppose that M is a connected, compact surface. Assume that $\mathrm{H}_1(M;\mathbb{Z})$ is isomorphic to $\pi_1(M)$ and M has odd Euler characteristic. Which of the standard surfaces could M be homeomorphic to? Justify your answer.

8. Let T^2 be the torus and K be the Klein bottle.

(*a*) Describe a 2-sheeted covering $q\colon T^2 \longrightarrow K$.

(*b*) Describe the resulting homomorphisms $q_*\colon \mathrm{H}_n(T^2;\mathbb{Z}) \to \mathrm{H}_n(K;\mathbb{Z})$ on homology groups.