DEPARTMENT OF MATHEMATICS

Topology Preliminary Examination
June 2, 2017

---

1 Let A and B be compact subsets of a topological group G. Show that the product $AB = \{ab \mid a \in A, b \in B\}$ is a compact subset of G.

2 Let X be locally compact, non-compact, Hausdorff space and let $f : X \longrightarrow Y$ and $g : X \longrightarrow Z$ be dense embeddings into compact Hausdorff spaces Y and Z. Suppose that $Z - g(X) = \{x_\infty\}$ is a single point.

(a) Show that there exists a unique map $q : Y \longrightarrow Z$ such that $q \circ f = g$.

(b) Show that q is a quotient map.

3 The set $\mathrm{M}_n(\mathbf{R})$ of $n \times n$-matrices can be identified with $\mathbf{R}^{n^2}$ by using the n^2 entries as coordinates.

(a) Let $\mathrm{GL}_n(\mathbf{R}) \subset \mathrm{M}_n(\mathbf{R})$ be the subspace of nonsingular (invertible) matrices. Show that $\mathrm{GL}_n(\mathbf{R})$ is not connected.

(b) Let $\mathrm{SL}_n(\mathbf{R}) \subset \mathrm{GL}_n(\mathbf{R})$ be the subspace of matrices with determinant 1. Show that $\mathrm{SL}_n(\mathbf{R})$ is path connected. [*Hint*: You may use the fact that $\mathrm{SL}_n(\mathbf{R})$ is generated by the elementary matrices $E_{i,j}(r)$ for $i \neq j$, which look like the identity matrix but have value r placed in entry (i, j).]

4 Let X and Y be locally compact, Hausdorff spaces. If Y^X is the set of continuous maps from X to Y with the compact open topology, show that the composition of maps

$$Z^Y \times Y^X \to Z^X$$

is continuous.

5 For a space X, let CX be the *cone over* X, i.e. the quotient space

$$CX = X \times [0,1]/(X \times \{1\}).$$

Let $i : X \longrightarrow CX$ be the map $i(x) = (x, 0)$. Prove that a continuous map $f : X \longrightarrow Y$ is nullhomotopic if and only if there is a continuous map $g : CX \longrightarrow Y$ such that $g \circ i = f$.

(over)

6 Let X be a connected and locally path-connected space with base point $x_0 \in X$ and let $y_0 = (1, 1) \in T^2$.

(a) The torus T^2 is a topological group with group operation

$$m \colon T^2 \times T^2 \to T^2$$

given by pointwise multiplication of complex numbers. Show that $[(X, x_0), (T^2, y_0)]$ is a group with group operation μ given by

$$\mu(\alpha, \beta)(x) = m(\alpha(x), \beta(x))$$

(b) Show that the map

$$[(X, x_0), (T^2, y_0)] \to \mathrm{Hom}(\pi_1(X, x_0), \pi_1(T^2, y_0))$$

which assigns f_* to f is a monomorphism. [*Note* : $[(X, x_0), (T^2, y_0)]$ is the set of based homotopy classes of based maps from X to T^2].

7 Let x, y, and z be three distinct points in S^2. Find the fundamental group of the quotient space $X = S^2 / \{x, y, z\}$.

8 List all covers of $S^1 \times \mathbf{R}P^2$, up to equivalence, and for each cover give the corresponding subgroup of $\pi_1(S^1 \times \mathbf{R}P^2)$.