# Topology Preliminary Exam
# June, 2025

**On grading:** A necessary condition to pass this exam is to completely solve one point-set question and one of the more algebraic questions. An excellent exam will completely solve five problems and have some partial credit on the remaining questions.

**1.** Let $(Y, d)$ be a metric space.

a. For $p \in Y$, show that the function $f \colon Y \to \mathbb{R}$ defined by $f(x) = d(x, p)$ is continuous.

b. Show that if $Y$ is a connected metric space that contains at least two distinct points, then $Y$ is uncountable.

**2.** Embed in the plane the countable family of circles

$$C_n = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; \left(x - \tfrac{1}{n}\right)^2 + y^2 = \tfrac{1}{n^2}\right\}, \qquad n = 1, 2, \ldots$$

and let

$$H = \bigcup_{n=1}^{\infty} C_n \subset \mathbb{R}^2,$$

be the Hawaiian earring.

a. Show that $H$ is compact.

b. Recall that a space $X$ is locally simply-connected if it admits a basis of simply connected opens. Prove that $H$ is not locally simply-connected.

c. Conclude that $H$ cannot be given the structure of a 1-manifold.

**3.** a. Let $X$ be Hausdorff and let $A \subseteq X$ be a subspace. Show that if there is a retraction $r \colon X \to A$, then $A$ is closed in $X$.

b. Let $X$ be compact Hausdorff. Let $C$ be a nonempty closed subset of $X$, and let $U = X \setminus C$ be the complement of $C$. Show that the quotient $X/C$ is a one-point compactification of $U$.

**4.** Let $X$ be a space with finite fundamental group. Show that any map $f \colon X \to S^1$ must be null-homotopic.

**5.** Let $K$ be the Klein bottle and let $p, q$ be distinct points in $K$. Compute $\pi_1(K \setminus \{p, q\})$ with any choice of basepoint.

**6.** Let $M$ be the Mobius band obtained by identifying the edges of the rectangle $[0, 1] \times [0, 1]$ via $(0, t) \sim (1, 1 - t)$. Write $m$ for the image of $(0, 0)$. Recall that $\pi_1(M, m) \cong \mathbb{Z}$. For each integer $n \geq 1$ construct a connected $n$-sheeted covering

$$p_n \colon E_n \longrightarrow M$$

whose corresponding subgroup of $\pi_1(M, m)$ is $n\mathbb{Z}$.

**7.** Let $X = \Delta^3$ be the standard 3-simplex. That is,

$$X = \{(r_1, r_2, r_3, r_4) \in \mathbb{R}^4 \mid r_1 + r_2 + r_3 + r_4 = 1, r_i \geq 0 \text{ for } i = 1, 2, 3, 4\}.$$

Let $A \subset X$ be the union of the 1-dimensional faces of $X$. Compute the relative homology groups $H_n(X, A)$ for all $n$.

**8.** Begin with the wedge of two pointed circles $S^1 \vee S^1$. Let $\alpha\colon S^1 \to S^1 \vee S^1$ be the path composition of the path that first winds around the left circle $m$ times with the path that then winds around the right circle $n$ times. Define $X_{m,n}$ to be the space obtained by attaching a 2-cell to $S^1 \vee S^1$ along the map $\alpha$. Compute the integral homology groups $H_q(X_{m,n}; \mathbb{Z})$ in terms of $m$ and $n$.