Topology Preliminary Exam
Summer 2026

On grading: A necessary condition to pass this exam is to completely solve one point-set question and one of the more algebraic questions. An excellent exam will completely solve five problems and have some partial credit on the remaining questions.

(1) Let $\mathbb{N} = (1, 2, 3, \dots)$ have the cofinite topology $\mathcal{T}$, i.e. $U \subset \mathbb{N}$ is open if and only if $U = \varnothing$ or $\mathbb{N} \setminus U$ is finite.

If $\mathbb{R}$ has the Euclidean topology, show that any continuous function $f\colon (\mathbb{N}, \mathcal{T}) \to \mathbb{R}$ is constant.

(2) For topological spaces U and X, let U^X be the set of continuous maps from X to U with the compact open topology.

Show the map $\pi\colon U^X \times V^Y \to (U \times V)^{X\times Y}$ defined by $(f, g) \mapsto f \times g$ is continuous if X and Y are Hausdorff.

(3) Let G be a compact connected topological group and let U be an open neighborhood of the identity element $e \in G$.

(a) Show that there exists an open $V \subseteq U$ containing e and such that $V = V^{-1} := \{v^{-1} : v \in V\}$.

(b) Let $V^n := \{v_1 v_2 \cdots v_n : v_i \in V\}$.

Show that $H = \bigcup_{n\geq 1} V^n$ is an open subgroup of G.

(c) Recall that open subgroups of a topological group are closed.

Use this to conclude that there exists an natural number $N \geq 1$ such that $G = U^N$.

(4) Consider the real projective plane $\mathbb{R}P^2$ and the torus $T^2 = S^1 \times S^1$.

(a) Prove that every continuous map $\mathbb{R}P^2 \to T^2$ is homotopic to a constant map.

(b) Show that there is no covering map $T^2 \to \mathbb{R}P^2$.

(5) Let $Y = \big((S^1 \vee S^1) \times I\big) \,/\, \big((S^1 \vee S^1) \times \partial I\big)$. Compute $\pi_1(Y, y)$ for $y \in Y$.

(6) Let $X = \mathbb{R}P^2 \times \mathbb{R}P^2$. For $x \in X$, compute $\pi_1(X, x)$, describe the universal cover $\tilde{X}$, and describe the deck transformations of $\pi_1(X, x)$ on $\tilde{X}$.

(7) Let $P \subset \mathbb{R}P^2$ be the image of the equator in S^2 under the quotient $S^2 \to \mathbb{R}P^2$. Compute the homology groups $H_k(\mathbb{R}P^2, P)$ for all k.

(8) A map $S^n \to S^n$ is degree d if it induces the multiplication by d map on $H_n(S^n)$.

Suppose the two ends of $S^2 \times I$ are glued together via a map $S^2 \to S^2$ of degree d. Use the Mayer-Vietoris sequence to calculate the homology of the resulting space X.