Research Statement

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The focus of my research has been in the area of infinite-dimensional Lie algebras. In my Ph. D. thesis [11] and in [12], I examined problems relating to the representation theory of *affine Lie algebras* and their close relatives, the *Heisenberg algebras*. Since their introduction by V. Kac and R. Moody, affine Lie algebras and their highest weight representations have been studied extensively and have been shown to have many applications in such diverse areas as integrable systems, partition identities, special functions, combinatorics, vertex operator algebras, and physics. Currently, I am also working on problems relating to *n-point affine Lie algebras*, which are generalizations of affine Lie algebras. These infinite-dimensional Lie algebras, as well as the Virasoro algebra, are subclasses of the *affine Krichever-Novikov algebras*, which were introduced in [23]-[25]. These algebras relate to some fundamental problems in geometry, analysis and mathematical physics, such as the twenty-first of the twenty three Hilbert problems, which is more commonly known as the *Riemann-Hilbert problem* [35]. My work offers several directions for further research within representation theory, quantum groups, vertex operator algebras, combinatorics and mathematical physics.

1. Whittaker modules for Heisenberg and Affine Lie Algebras

In R. Block's classification [2] of all irreducible modules for the Lie algebra $\mathfrak{sl}_2$ of traceless 2×2 complex matrices, the irreducible $\mathfrak{sl}_2$-modules fall into three families: highest (lowest) weight modules, *Whittaker* modules, and a third family obtained by localization. This result illustrates the prominent role played by Whittaker modules in the representation theory of Lie algebras.

In my thesis [11], I investigated a particular class of representations for affine Lie algebras which resemble the Whittaker modules for finite-dimensional semisimple Lie algebras, and therefore I refer to them as *Whittaker modules*. My thesis is the first work to develop a theory of Whittaker modules for affine Lie algebras.

Let $\mathfrak{g}$ be a Lie algebra with triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ (see [30]), where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. For example, if $\mathfrak{g} = \mathfrak{gl}_n$, the Lie algebra of $n \times n$ matrices, then $\mathfrak{h}$ is the subalgebra of diagonal matrices and $\mathfrak{n}_+$ (respectively $\mathfrak{n}_-$) is the subalgebra of strictly upper (respectively strictly lower) triangular matrices. Examples of Lie algebras with triangular decomposition are the finite-dimensional semisimple Lie algebras, the affine Lie algebras, the Virasoro algebra, and Heisenberg algebras extended by derivations. Let $\mathcal{U}(\mathfrak{n}_+)$ be the (universal) enveloping algebra of $\mathfrak{n}_+$ and let $\eta : \mathcal{U}(\mathfrak{n}_+) \to \mathbb{C}$ be an algebra homomorphism such that $\eta|_{\mathfrak{n}_+} \neq 0$. A *Whittaker* module for $\mathfrak{g}$ of type η is any $\mathfrak{g}$-module V that is generated by a vector w (known as a *Whittaker vector*) which is an eigenvector for $\mathfrak{n}_+$ with eigenvalues given by η. Thus $xw = \eta(x)w$ for all $x \in \mathfrak{n}_+$. (The excluded case $\eta|_{\mathfrak{n}_+} = 0$ leads to the well-known highest weight modules).

The class of Whittaker modules for an arbitrary finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$ was defined by B. Kostant in [22]. He termed these modules Whittaker because of their connections with Whittaker equations in number theory. In [37], N. Wallach gave new proofs of Kostant's results in the case $\mathfrak{g}$ is the product of complex Lie algebras isomorphic to $\mathfrak{sl}_\mathfrak{n}$. E. McDowell [28], and D. Miličić and W. Soergel [29] studied a category of modules for an arbitrary finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$ which includes the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ as well as those Whittaker modules W which are locally finite over the center $\mathcal{Z}(\mathfrak{g})$ of the enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ (i.e. $\mathcal{Z}(\mathfrak{g})v$ is finite-dimensional for each $v \in W$). Recently, certain categories of generalized Whittaker modules have been shown to have connections with the representation theory of finite W-algebras in the work of A. Premet [33], and J. Brundan, A. Kleshchev [5]. In the quantum setting, M. Ondrus classified Whittaker modules for the quantum enveloping algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ in [31].

In my thesis [11] I construct Whittaker type modules for affine Lie algebras using parabolic induction. Lie algebras with triangular decomposition come equipped with an *involution* $\sigma : \mathfrak{n}_+ \to \mathfrak{n}_-$. A Lie subalgebra $\mathfrak{p}$ is called a parabolic subalgebra if $\mathfrak{h} \subset \mathfrak{p}$ and $\mathfrak{p} + \sigma(\mathfrak{p}) = \mathfrak{g}$. In [15], V. Futorny classified the parabolic subalgebras of affine Lie algebras and showed that every parabolic subalgebra $\mathfrak{p}$ has a decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{r}$, where $\mathfrak{l}$ is called the *Levi factor* of $\mathfrak{p}$, and $\mathfrak{l}$ is either a finite-dimensional reductive Lie algebra or $\mathfrak{l}$ contains an infinite-dimensional Heisenberg algebra. The parabolic subalgebras of the first type are called *standard* if they contain $\mathfrak{h} \oplus \mathfrak{n}_+$ and they have motivated my work described in Section 1.1. The parabolic subalgebras of the second type inspired my work described in Sections 1.2 and 1.3.

1.1. **Whittaker Modules Induced from Standard Parabolic Subalgebras.** For this section, let $\mathfrak{g}$ be an affine Lie algebra (see [20]). In [11], given a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ and an algebra homomorphism $\eta : \mathcal{U}(\mathfrak{n}_+) \to \mathbb{C}$ such that $\eta|_{\mathfrak{n}_+} \neq 0$ and η is zero on at least one of the generators of $\mathfrak{n}_+$, I associate to η a standard parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$. I study Whittaker modules $M(W)$ of type η for $\mathfrak{g}$, which are constructed by inducing over the subalgebra $\mathfrak{p}$ starting from irreducible Whittaker modules W for the finite-dimensional reductive Levi factor $\mathfrak{l}$ of $\mathfrak{p}$. I show that $M(W)$ has a unique irreducible quotient $L(W)$. For the rest of this section, I specialize to the case $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$. The affine Lie algebra $\widehat{\mathfrak{sl}_2}$, also denoted $A_1^{(1)}$, is the "smallest" infinite-dimensional Kac-Moody Lie algebra, but perhaps the most important. Every parabolic subalgebra with a finite-dimensional Levi factor for $\mathfrak{g}$ (for a fixed set of simple roots π) contains $\mathfrak{h} \oplus \mathfrak{n}_+$, and the semisimple part of its Levi factor is isomorphic to $\mathfrak{sl}_2$. In this case, I was able to show the following result classifying certain Whittaker modules for $\widehat{\mathfrak{sl}_2}$:

Theorem 1. [11] *Assume $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$. Let $\eta : \mathcal{U}(\mathfrak{n}_+) \to \mathbb{C}$ be an algebra homomorphism such that $\eta|_{\mathfrak{n}_+} \neq 0$ and $\eta(e_\alpha) = 0$ for some generator e_α of $\mathfrak{n}_+$, and let $\mathfrak{p}$ be the standard parabolic subalgebra associated to η. Let V be an irreducible Whittaker module of type η for $\mathfrak{g}$. Then there exists an irreducible Whittaker module W for the finite-dimensional Levi factor $\mathfrak{l}$ of the subalgebra $\mathfrak{p}$ such that $V \cong L(W)$.*

In [11], I also establish an irreducibility criterion for the modules $M(W)$ for $\mathfrak{g} = \widehat{\mathfrak{sl}_2}$.

1.2. **Whittaker Modules for Heisenberg Algebras.** Heisenberg algebras arise naturally in the following context: Let $P = \mathbb{C}[t_j]_{j \in \mathbb{Z}_{>0}}$ be the polynomial algebra over $\mathbb{C}$ in infinitely many commuting variables t_j, $j \in \mathbb{Z}_{>0}$. Consider the following linear operators on P: L_j acts as multiplication by t_j, $\dfrac{\partial}{\partial t_j}$ acts as the partial derivative with respect to t_j, and 1 acts as the identity operator. The linear span $\mathfrak{t}$ of all these operators is an infinite-dimensional Heisenberg algebra with $[L_i, L_j] = [\dfrac{\partial}{\partial t_i}, \dfrac{\partial}{\partial t_j}] = 0$, $[\dfrac{\partial}{\partial t_i}, L_j] = \delta_{ij}1$, and $[\mathfrak{t}, 1] = 0$. Moreover, $\mathfrak{t}$ is $\mathbb{Z}$-graded (as a Lie algebra). The Heisenberg algebras which I consider in [11, 12] are infinite-dimensional and can be realized as above. Let $\mathfrak{t} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{t}_i$ be an infinite-dimensional Heisenberg algebra of this type with a one-dimensional center $\mathfrak{t}_0 = \mathbb{C}c$. If V is an irreducible

$\mathfrak{t}$-module, then c acts as a scalar, called the *level*. In [11, 12], I describe the irreducible Whittaker modules for $\mathfrak{t}$. These modules are not $\mathbb{Z}$-graded as $\mathfrak{t}$-modules. All my results are also valid for any finite-dimensional Heisenberg algebra with some minor modifications in the definitions. From the Whittaker $\mathfrak{t}$-modules of level one, I obtain irreducible Whittaker modules for Weyl algebras.

In my dissertation and in [12] I also study the irreducible Whittaker modules for the Lie algebra $\widetilde{\mathfrak{t}}$ obtained by extending $\mathfrak{t}$ by a degree derivation d. The central element c of $\widetilde{\mathfrak{t}}$ again acts by a scalar (denoted by a and called the *level*) on any irreducible $\widetilde{\mathfrak{t}}$-module. I show that for any $a \in \mathbb{C}$ and any algebra homomorphism $\eta : \mathcal{U}(\widetilde{\mathfrak{t}}^{+}) \to \mathbb{C}$ such that $\eta|_{\widetilde{\mathfrak{t}}^{+}} \neq 0$ and $\eta|_{\mathfrak{t}_i} \neq 0$ for infinitely many $i \in \mathbb{Z}_{>0}$ if $a \neq 0$, there exists a unique (up to isomorphism) irreducible Whittaker $\widetilde{\mathfrak{t}}$-module $\widetilde{L}_{\eta,a}$ of type η and level a.

1.3. **Imaginary Whittaker Modules for Affine Lie Algebras.** *Loop modules* for an affine Lie algebra $\mathfrak{g}$ are modules induced over a parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{l} = \mathfrak{t} + \mathfrak{h}$, where $\mathfrak{t}$ is an infinite-dimensional Heisenberg subalgebra, and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, starting from irreducible $\mathbb{Z}$-graded $\mathfrak{t}$-modules. The central element of $\mathfrak{g}$ then acts as a scalar called the level of the module. Integrable loop modules of level zero were studied in [7, 8, 9, 10], but arbitrary loop modules of level zero are still not completely classified. Loop modules of non-zero level are also called imaginary Verma modules, and they were studied in [16]. Analogues of imaginary Verma modules have also been constructed for the quantum group $\mathcal{U}_q(\mathfrak{g})$ of a non-twisted affine Lie algebra $\mathfrak{g}$ in [18], and for the extended affine Lie algebra $\mathfrak{sl}_2(\mathbb{C}_q)$ in [13].

In my thesis and in [12] I use my results from Section 1.2 to construct a new class of modules for non-twisted affine Lie algebras, which I call *imaginary Whittaker modules*, as they are constructed by inducing over the same parabolic subalgebra as the imaginary Verma modules or loop modules, but with the root vectors corresponding to the positive imaginary roots acting in a non-zero fashion. More specifically, let $\dot{\mathfrak{g}}$ be a finite-dimensional complex simple Lie algebra, and fix a triangular decomposition $\dot{\mathfrak{g}} = \dot{\mathfrak{n}}_- \oplus \dot{\mathfrak{h}} \oplus \dot{\mathfrak{n}}_+$ of $\dot{\mathfrak{g}}$, where $\dot{\mathfrak{h}} \subseteq \dot{\mathfrak{g}}$ is a Cartan subalgebra. Let $\mathfrak{g}$ be the non-twisted affine Lie algebra associated with $\dot{\mathfrak{g}}$. Therefore $\mathfrak{g} = (\dot{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}d$, where the element c is central and d is a derivation. Set $\widetilde{\mathfrak{t}} = (\dot{\mathfrak{h}} \otimes t\mathbb{C}[t]) \oplus (\dot{\mathfrak{h}} \otimes t^{-1}\mathbb{C}[t^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}d$. The subalgebra $\widetilde{\mathfrak{t}}$ motivated the definitions in Section 1.2, and so one can apply all the results on Whittaker modules from Section 1.2 to $\widetilde{\mathfrak{t}}$. Set $\mathfrak{n}^+ = \dot{\mathfrak{n}}_+ \otimes \mathbb{C}[t, t^{-1}]$ and $\mathfrak{p} = (\widetilde{\mathfrak{t}} \oplus \dot{\mathfrak{h}}) \oplus \mathfrak{n}^+$. The subalgebra $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ and it contains the infinite-dimensional Heisenberg algebra $\mathfrak{t}$. Let $\lambda \in (\dot{\mathfrak{h}} \oplus \mathbb{C}c)^*$. Set $\widetilde{\mathfrak{t}}^{+} = \dot{\mathfrak{h}} \otimes t\mathbb{C}[t]$, and let $\eta : \mathcal{U}(\widetilde{\mathfrak{t}}^{+}) \to \mathbb{C}$ be an algebra homomorphism such that $\eta|_{\widetilde{\mathfrak{t}}_{+}} \neq 0$ and $\eta|_{\mathfrak{t}_i} \neq 0$ for infinitely many $i \in \mathbb{Z}_{>0}$ if $\lambda(c) \neq 0$, where now $\mathfrak{t}_i = \dot{\mathfrak{h}} \otimes t^i$ for $i \neq 0$. View $\widetilde{L}_{\eta,\lambda(c)}$ as a $\mathcal{U}(\mathfrak{p})$-module by letting $hw = \lambda(h)w$ for all $w \in \widetilde{L}_{\eta,\lambda(c)}$, $h \in \dot{\mathfrak{h}} \oplus \mathbb{C}c$, and $\mathfrak{n}^+$ act trivially on $\widetilde{L}_{\eta,\lambda(c)}$. Set $V_{\eta,\lambda} = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \widetilde{L}_{\eta,\lambda(c)}$. The $\mathfrak{g}$-module $V_{\eta,\lambda}$ is called an *imaginary Whittaker module of type* (η, λ) *for* $\mathfrak{g}$. My main result is the following:

Theorem 2. [11, 12] *Let* $\mathfrak{g} = (\dot{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathbb{C}c \oplus \mathbb{C}d$ *be a non-twisted affine Lie algebra and let* $\lambda \in (\dot{\mathfrak{h}} \oplus \mathbb{C}c)^*$.

(i) *If* $\lambda(c) \neq 0$, *then the imaginary Whittaker module* $V_{\eta,\lambda}$ *is irreducible as a* $\mathcal{U}(\mathfrak{g})$*-module.*
(ii) *If* $\lambda(c) = 0$, *then* $V_{\eta,\lambda}$ *has a unique irreducible quotient* $L_{\eta,\lambda}$.

Theorem 2(i) is an analogue of a similar result for imaginary Verma modules [17, Prop. 5.8].

1.4. **Directions for Further Work.** My dissertation is a first attempt to develop a theory of Whittaker modules for affine Lie algebras. Several natural problems arise from this work for future investigation. Theorem 1 classifies certain irreducible Whittaker modules for $\widehat{\mathfrak{sl}_2}$. In the future, I hope to describe the irreducible Whittaker modules for η non-zero on all generators of $\mathfrak{n}_+$ in the affine case and thus completely classify all irreducible Whittaker modules for $\widehat{\mathfrak{sl}_2}$. I also hope to investigate whether Theorem 1 can be carried over to other affine Lie algebras, and to classify the irreducible Whittaker modules for arbitrary affine $\mathfrak{g}$. One interesting question is what are the annihilator ideals in the enveloping algebra of an arbitrary affine $\mathfrak{g}$ of the irreducible Whittaker modules $L(W)$ described in Section 1.1. Another direction would be to study analogous modules for loop algebras and the Virasoro algebra.

I also hope to construct analogues of the imaginary Whittaker modules for twisted affine Lie algebras and other infinite-dimensional Lie algebras, for example, extended affine Lie algebras, and describe the submodule structure and irreducible quotients of the imaginary Whittaker modules of level zero.

In [31, 32], M. Ondrus classified Whittaker modules for the quantum enveloping algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$, and studied their tensor products with finite-dimensional modules for $\mathcal{U}_q(\mathfrak{sl}_2)$. However, the quantum Serre relations used to define the enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ of a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$ imply that if $\mathcal{U}^+$ is the subalgebra of $\mathcal{U}_q(\mathfrak{g})$ analogous to $\mathcal{U}(\mathfrak{n}_+)$ and $\eta : \mathcal{U}^+ \to \mathbb{C}$ is an algebra homomorphism, then η has to vanish on at least one of the generators of $\mathcal{U}^+$ if $\mathfrak{g} \neq \mathfrak{sl}_2$. In [34], A. Sevostyanov showed that $\eta(e_\alpha) \neq 0$ for all $\alpha \in \pi$ is possible; however, his result was for a different quantum algebra, namely the topological Hopf algebra $\mathcal{U}_h(\mathfrak{g})$ over $\mathbb{C}[[h]]$. It would be interesting in both the finite-dimensional, and the affine case to examine conditions under which Whittaker and imaginary Whittaker modules can be deformed to representations of the corresponding quantum group.

2. n-point Affine Lie Algebras

In ongoing joint work with Michael Lau, we are studying *bosonic* and *fermionic* representations for *n-point affine Lie algebras.* Let $\dot{\mathfrak{g}}$ be a finite-dimensional complex simple Lie algebra. Since the ring $\mathbb{C}[t, t^{-1}]$ of Laurent polynomials is the ring of rational functions on the Reimann sphere $\mathbb{C} \cup \{\infty\}$ with poles allowed only in $\{\infty, 0\}$, there is a natural generalization of the loop algebra construction $\dot{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$, by considering any algebraic curve L of genus g with a fixed subset P of n distinct points, and the ring R of meromorphic functions on L with poles allowed only in P, and forming the infinite-dimensional Lie algebra $\dot{\mathfrak{g}} \otimes_{\mathbb{C}} R$.

Central extensions are obtained by enlarging the center of the original infinite-dimensional Lie algebras and they are an important tool in representation theory, as they enrich the collection of available representations. In the case of genus 0, M. Bremner [4] determined the *universal central extension* of the loop algebra $\dot{\mathfrak{g}} \otimes_{\mathbb{C}} R$, where $R = \mathbb{C}[t, (t-a_1)^{-1}, \ldots, (t-a_{n-1})^{-1}]$ with $a_i \in \mathbb{C}$, and showed that the center has dimension $n-1$. The Lie algebras obtained in this way are called *n-point affine Lie algebras.* In [1], G. Benkart and P. Terwilliger realized the three-point $\mathfrak{sl}_2$ loop algebra as the *tetrahedron algebra* given in [19], and gave a presentation by generators and relations of its universal central extension.

Fock space is a vector space spanned by monomials corresponding to the possible energy states of particles in some physical system. It is called *bosonic* or *fermionic* according to the occupancy statistics that the particles satisfy. Various Lie algebras have natural actions on Fock spaces, called *bosonic* or *fermionic representations.* For affine Lie algebras, these include the *oscillator/spinor representations* [14, 21], as well as the *vertex operator representations* first given by J. Lepowsky and R. L. Wilson in [27]. In [26], M. Lau gave a uniform construction of bosonic and fermionic modules for a one-dimensional central extension $\widetilde{g}$ of any Lie algebra $\mathfrak{g}$ over a field of characteristic 0. Fermionic representations for the two-point affine Krichever-Novikov algebras were described in [36]. Bosonic realizations of the four-point $\mathfrak{sl}_2$ loop algebra have been obtained by B. Cox in [6]. We have constructed a fermionic realization of this Lie algebra and in the future we hope to generalize our construction to the n-point case. We are also interested in studying connections with vertex operators and combinatorial identities.

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