Research Statement

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1 Graduate Research

My research uses combinatorial structures to study problems in algebraic geometry. I work on the classification of curves that cannot be embedded in a $K3$ surface. This directly relates to surjectivity of the Wahl map, which is a special case of the better-known Gauss map. We reframe the surjectivity problem to one in graph theory by using graph curves. I earned the Ed Enochs Graduate Scholarship in Algebra for my research at the University of Kentucky in 2021.

1.1 The Wahl Map

The moduli space of curves M_g of genus g has some properties for small values of g that do not generalize for higher values of g. For instance, M_g is uniruled if $g \leq 9$ or $g = 11$, but this does not hold in general. The study of why M_g is not uniruled for all g is related to $K3$ surfaces. Therefore, knowing if an algebraic curve can be embedded in a $K3$ surface can identify important structures and properties of that curve. The Wahl map $\mathbb{W}$ gives a necessary condition for an algebraic curve to be embedded in a $K3$ surface [Wah87]:

Theorem 1 (Wahl)**.** *If a smooth algebraic curve C can be embedded in a $K3$ surface, then $\mathbb{W}$ is not surjective on C.*

We will define the Wahl map for graphs later on. Ciliberto, Harris, and Miranda were some of the first to study when the Wahl map is surjective [CHM88], proving the following result:

Theorem 2 (Ciliberto, Harris, Miranda)**.** *If C is a general curve of genus $g \geq 10$ and $g \neq 11$, then the Wahl map is surjective on C.*

Notice that this classification is only for *general* curves and not for all curves. Our goal is to expand these results with the aid of graph theory.

1.2 Specialization to Graphs

We use graphs to represent curves in two steps. First, we generalize from the case of smooth curves to singular. Second, we specialize to cubic graphs. This specialization starts with a graph curve. A graph curve is a curve whose connected components are lines and no three components meet at any singularity. To transform a graph curve to a graph, we take the dual in the following way. Each line in a graph curve corresponds to a vertex in the dual graph. Vertices are connected in the dual graph if the original lines intersect in the graph curve. Since exactly two components meet at any singularity, we are guaranteed to have only two vertices connected by a single edge. See Figure 1 for a demonstration of this dualization.

The dual graph lends itself to well understood ideas from graph theory like graph coloring, connectedness, and planarity. My research considers the Wahl map as defined for cubic (i.e. 3-regular or trivalent) graphs [CF92]. A graph must be cubic to uniquely determine the corresponding

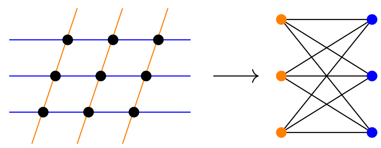

Figure 1: The orange, vertical components of the graph curve are parallel, so they do not have edges between them in the dual. Similarly, the blue, horizontal components of the graph curve are parallel and do not have edges between them in the dual. Every place that an orange component meets a blue component, we have a corresponding edge in the dual.

graph curve. If the Wahl map is surjective on a graph curve, then it is also surjective on the family of curves that degenerate to that graph curve. Hence it suffices to consider surjectivity for cubic graphs.

The Wahl map on cubic graphs

$$\mathbb{W}_G : \bigwedge^2 H_1(G,k) \to Cat_0(G,k) \oplus Cat_1(G,k)$$

takes in a pair of cycles from a cubic graph G and returns a vector of values on the vertices and edges. Because the codomain is a direct sum, $\mathbb{W}_G$ decomposes as $\mathbb{W}_G = V \oplus E$, where the codomain of V is generated by the vertices of the graph and the codomain of E is generated by the edges of the graph. The important take-away is that we can consider our problem in two pieces:

1. When is V surjective?
2. When is $\mathbb{W}_G|_{kerV}$ surjective?

1.3 My Results

We have determined the necessary and sufficient graphic criteria to answer the first question. We have the necessary conditions from Miranda [Mir91]. Note that a graph is 3-connected if you can remove any two vertices from the graph and stay connected.

Theorem 3 (Miranda)**.** *Let G be a cubic, 3-connected graph. If V_G is surjective, then G is non-planar.*

We found that surjectivity of V is preserved under refinement in the following way:

Theorem 4 (Hanson)**.** *Let G and H be cubic graphs. Suppose G is 3-connected and contains a subgraph H' that is a refinement of H. If V_H is surjective, then V_G is also surjective.*

The necessary conditions from Theorem 3 are also sufficient:

Theorem 5 (Hanson)**.** *Let G be a cubic, 3-connected graph. If if G is non-planar, then V_G is surjective.*

Recall that the graph must be cubic to apply the Wahl map, so the 3-connectedness and non-planarity are the surjectivity conditions. The key to the argument is that for $G = K_{3,3}$ we find that $\mathbb{W}_G$ is surjective. For cubic graphs, containing a subdivision of $K_{3,3}$ is equivalent to being non-planar [Men30] so by Theorem 4 and surjectivity on $K_{3,3}$, we prove Theorem 5.

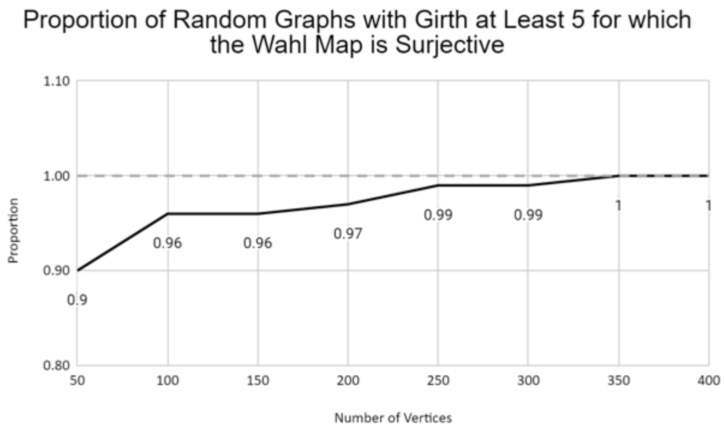

Figure 2: This chart depicts the proportion of random cubic graphs which are surjective under the Wahl map. The examples are of 50, 100, 150, 200, 250, 300, 350, and 400 vertices with 100 sample graphs of each size.

It remains to determine for which graphs G the map $\mathbb{W}_G$ restricted to $\ker(V)$ is surjective. We've discovered that this relates to the girth of G. The girth of a graph is the minimum length that any cycle can have in the graph.

Theorem 6 (Hanson). *If $\mathbb{W}_G$ is surjective, then girth$(G) \geq 5$.*

The heart of the argument for Theorem 6 is that there are not enough cycles that pass through 3- or 4-cycles to be surjective locally, let alone on the entire graph.

1.4 Future Work

In addition to classifying the graphs for which $\mathbb{W}_G$ is surjective, we want to know how common these graphs are. For instance, if we generate a random, cubic graph G, how likely is it that $\mathbb{W}_G$ is surjective? This can be answered probabilistically. From data (Figure 2) produced by a program that I wrote in SageMath, we have the following conjecture:

Conjecture 1 (Hanson). *Let F_G be the event that $\mathbb{W}_G$ is surjective. For a random cubic graph G with girth$(G) \geq 5$ and n vertices, we have*

$$\lim_{n\to\infty} P(F_G) = 1.$$

Since cubic graphs are almost always 3-connected [RW92], there is a high probability that this criteria is met. Similarly, as the number of vertices grows, the chance that a graph is planar goes to zero. As a result, Conjecture 1 tells us that we have found a class of graphs for which the Wahl map is surjective with high probability, so my next goal is the following:

Project 1. *Prove Conjecture 1.*

We have also been studying $\mathbb{W}_G$ on graphs of varying gonality. Gonality offers a means for categorizing graphs as we study $\mathbb{W}_G$. It is important to note that gonality of curves is an upper bound for gonality of graphs, not an equality. However, we can still examine results about the Wahl map on k-gonal curves to gain insight into $\mathbb{W}_G$ on k-gonal graphs. For instance, it has already been shown that the Wahl map is not surjective for *curves* of gonality $k = 2, 3$ [CM92, Wah90] and $k = 4$ [Bra95, Bra97]. Also, Ciliberto and Lopez proved the following for *curves* [CL02]:

Theorem 7 (Ciliberto, Lopez). *Let C be a general k-gonal curve of genus $g \geq 12$. Then $\mathbb{W}_C$ is surjective if*

(i) $k = 5$ *and* $g \geq 15$*;*

(ii) $k = 6$ *and* $g \geq 13$*;*

(iii) $k \geq 7$.

Project 2. *For each of the following pairs (k, g), I want to find examples where $\mathbb{W}_C$ is surjective for a curve C of gonality k and genus g or show that no such example exists, where the values of k and g are*

(i) $k = 5$ *and* $g = 12, 13, 14$*, or*

(ii) $k = 6$ *and* $g = 12$.

It would be interesting to see how these cases compare to their graphic counterparts.

Project 3. *For $k \leq 4$, I want to find examples where $\mathbb{W}_G$ is surjective for a graph G of gonality k or show that no such example exists.*

I am similarly interested in the classification of graphs of a given gonality. The gonality 2 [Cha12] and 3 [ADM$^+$19] cases have been studied, but there are many are left to explore.

The Wahl map is a special case of the Gauss map. While surjectivity of the Gauss map is well-studied, there are still several open questions about the injectivity of the Gauss map. The kernel and injectivity of the Gauss map have connections to smoothness of curves, embeddings in $\mathbb{P}^3$, and self-correspondences on a curve, each depending on the projective variety and line bundle used to define the map [Wah92], resulting in a number of cases still to consider.

Project 4. *For which line bundles is the Gauss map injective?*

To explore injectivity, it is worth determining if there is a combinatorial version of the Gauss map like the graph theoretic version of the Wahl map.

2 Research with Undergraduates

In addition to my primary research, I hope to continue working on projects that encourage junior collaborators, especially undergraduates. While I reframe advanced algebraic problems into combinatorial ones in my own research, this research also provides opportunities to make problems more attainable.

2.1 Sliding-Block Puzzles

I have worked with undergraduate students at the University of Kentucky to study the (15+4)-puzzle by reframing the problem into a graph theoretic one. This puzzle is a generalization of the better-known 15-puzzle. The 15-puzzle is a 4×4 grid of 15 tiles with one empty space. These tiles can shift up, down, left, or right within the bounds of a frame. Figure 3 is a picture of the 15-puzzle. The (15+4)-puzzle has five 2×2 grids of tiles instead of one 4×4 grid. The 2×2 grids are joined at a common corner, and tiles can move within and between grids as in the 15-puzzle. Figure 3 provides a visual of this 3-dimensional puzzle. The biggest difference between

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

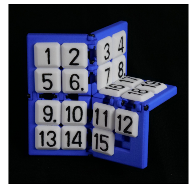

Figure 3: These are the 15-puzzle and (15+4)-puzzle. The (15+4)-puzzle is created by Henry Segerman of Oklahoma State University.

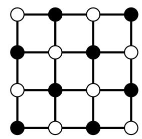

Figure 4: This is the graph of the 15-puzzle with bipartite coloring (black and white).

the 15-puzzle and the (15+4)-puzzle is that when a tile moves one full cycle around the puzzle, it returns to its original spot rotated 90 degrees. Therefore, in addition to tracking location of the tiles within the puzzle, it is important to record how many times the tile has been rotated.

2.2 Solvability of the 15-Puzzle

To study if the 15-puzzle is solvable, Richard Wilson converted the puzzle into a graph where each vertex corresponds to the possible tile locations and an edge indicates if a tile can slide between two locations [Wil74]. You can see the graph of the 15-puzzle in Figure 4.

Notice that this graph of the 15-puzzle is bipartite because we can color every vertex using two colors and no vertices of the same color are adjacent. Wilson found that this is the key to finding the solvable tile arrangements. He used this to generalize to the following result [Wil74]:

Theorem 8 (Wilson)**.** *Let G be a 2-vertex-connected, simple graph on $n+1$ vertices that is not a cycle or the theta graph. If G is bipartite, then the solvable set is the set of even permutation of n tiles, the alternating group A_n. Otherwise, any permutation of the n tiles is solvable, so the solvable set is the symmetric group S_n.*

2.3 Our Results

Theorem 8 is insufficient for the (15+4)-puzzle because tiles rotate. We added an edge weight to indicate if tiles rotate and by how much. The weights are $k \bmod m$, where k is the number of rotations by $360/m$ degrees. For instance, Figure 5 is the graph of the (15+4)-puzzle. It is important to note that the graph of the (15+4)-puzzle is not bipartite, but it does have the related property of being twist bipartite, which is relevant to Theorem 9.

Including rotations in the graph changes the set of solvable permutations to be a subgroup of the generalized symmetric group $S(m,n)$ instead of S_n. This group has elements of the form $(\vec{r},\sigma)$, where r_i is the number of rotations on tile i and σ is the permutation of the tiles. In the case of the $(15+4)$-puzzle, there are 19 tiles, and there are 4 possible orientations for each tile. Therefore, the solvable states of the puzzle are a subgroup of $S(4,19)$. We generalized to a larger class of puzzles with the following result:

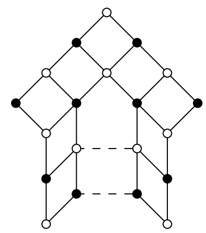

Figure 5: This is the graph of the (15+4)-puzzle with the dashed edges having weight 1 mod 4. This weight indicates that crossing the edge once produces one rotation of $360/4 = 90$ degrees. If a tile crosses the dashed edges from left to right, then it rotates 90 degrees in one direction. If a tile crosses the dashed edges from right to left, it rotates 90 degrees in the other direction.

Theorem 9 (Garcia, Hanson, Jensen, Owen)**.** *Under certain mild assumptions about divisibility and connectivity of the graph, the solution set for any graph is one of the following:*

1. $\{(\vec{r}, \sigma) \mid \sigma \in A_n\}$ *if the graph is bipartite,*
2. $\{(\vec{r}, \sigma) \in S(m,n) \mid \sum_i r_i \equiv sgn(\sigma) \pmod 2\}$ *if the graph is twist bipartite, or*
3. $S(m,n)$ *otherwise.*

2.4 Future Work

There are further generalizations of this type of puzzle. For instance, consider a sliding-block puzzle where the tiles are cubes and tile rotations are rotations of the cube. Then rotations are no longer cyclic. This presents a natural continuation of the early slide-block puzzle research.

Project 5. *What is the solution set for a puzzle with non-cyclic tile rotations?*

Seeing how well the students did with the graph theoretic techniques, I would like to use graphs in other undergraduate research projects, like graph gonality. Since gonality of graphs can be different than gonality of corresponding curves, knowing more about graph gonality can help clarify the relationship between the two. While calculating graph gonality is generally NP-hard, highly symmetric graphs are much easier to work with by reducing the number of cases to consider.

Project 6. *What can be said about the graph gonality for families of graphs with many symmetries like the generalized Petersen graphs?*

In a similar vein, there are algorithms to test if a given graph has gonality 2 or 3. There is potential to study these algorithms and see if they can be optimized or generalized to higher gonality.

Project 7. *Build an algorithm to check if a given graph has gonality 4.*

In either project there is an opportunity to collaborate with computer science or software engineering students to generate and test examples since graph gonality is calculated algorithmically.

2.5 Motivation for Working with Undergraduates

Any chance to work with undergraduates on research is an opportunity to foster their enthusiasm for math and to pursue it professionally. Plus, research allows students to learn more about the many areas of math and find where their interests lie.

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