RESEARCH STATEMENT

MARIE MEYER

My primary research interests lie in algebraic and geometric combinatorics, which is a subfield of discrete mathematics that applies techniques from abstract algebra and geometry to combinatorial objects. I currently study combinatorial objects through the lens of Ehrhart theory, which I describe in the first section of this statement. In the next two sections I present my contributions to the field and include open questions leading to future work. Finally I highlight ways to extend my research to involve undergraduate students.

1. Introduction to Ehrhart Theory

The combinatorial objects at the center my research are polytopes. Polytopes are generalizations of the familiar polygons and polyhedra, polytopes of dimension two and three, respectively. As geometric objects with combinatorially interesting properties, polytopes are shown to have important applications in diverse areas such as topology, optimization, and quantum physics, to name a few. Formally, a *lattice polytope* of dimension d is the convex hull of finitely many points in the lattice $\mathbb{Z}^n$, which together affinely span a d-dimensional hyperplane of $\mathbb{R}^n$. Whenever a polytope of dimension d is formed by the convex hull of exactly $d+1$ vertices, we call the polytope a *simplex*. In the realm of polytopes, reflexive polytopes are a particularly interesting class first introduced in [2]. A lattice polytope $\mathcal{P}$ is called *reflexive* if it contains the origin in its interior, and its polar dual, denoted $\mathcal{P}^*$, is a lattice polytope. The *polar dual* of a polytope is $\mathcal{P}^* := \{x \in \mathbb{R}^d \mid x \cdot y \leq 1 \text{ for all } y \in \mathcal{P}\}$.

Ehrhart theory was developed to study discrete properties of polytopes, one of them being the lattice point count of a polytope and its dilates. For a positive integer t, the t^{th} *dilate* of $\mathcal{P}$ is given by $t\mathcal{P} := \{tp \mid p \in \mathcal{P}\}$. We recover these dilates of $\mathcal{P}$ by applying a technique called *coning over the polytope*. Given $\mathcal{P} = \text{conv}\{v_1, \ldots, v_n\} \subseteq \mathbb{R}^d$, lift these vertices into $\mathbb{R}^{d+1}$ by appending 1 as their last coordinate to define the points $w_1 = (v_1, 1), \ldots, w_n = (v_n, 1)$. The *cone over* $\mathcal{P}$ is

$$\text{cone}(\mathcal{P}) = \{\lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_n w_n \mid \lambda_1, \lambda_2, \ldots, \lambda_n \geq 0\} \subseteq \mathbb{R}^{d+1}\,.$$

For each positive integer t we can intersect cone$(\mathcal{P})$ with the hyperplane $x_{d+1} = t$ to obtain a copy of $t\mathcal{P}$. Now to each polytope $\mathcal{P}$ we associate a function which counts the number of lattice points in the t^{th} dilate of $\mathcal{P}$. This function is the *lattice point enumerator*, denoted $L_{\mathcal{P}}(t)$, and it is know to be a polynomial in t. The *Ehrhart series* of a polytope is the generating function of $L_{\mathcal{P}}(t)$. A result of Ehrhart in [9] shows we can write the generating function as a rational function of the form

$$\text{Ehr}_{\mathcal{P}}(z) = 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t = \frac{h_d^* z^d + h_{d-1}^* z^{d-1} + \cdots + h_1^* z + h_0^*}{(1-z)^{d+1}}.$$

The coefficients of the numerator, $h^*(\mathcal{P}) = (h_0^*, h_1^*, \cdots, h_d^*)$, is called the h^*-*vector*, also known as the δ-*vector*, of $\mathcal{P}$.

The h^*-vector is a well-studied object in Ehrhart theory as it admits combinatorial interpretations that reveal information about the polytope. For instance, the *normalized volume* of the polytope is given by $\sum_{i=0}^d h_i^*$. The *Euclidean volume* can be recovered by dividing the normalized volume by $d!$. Stanley proved the h^*-vector of a convex lattice polytope satisfies $h_0^* = 1$ and $h_i^* \in \mathbb{Z}_{\geq 0}$, which is found in [20]. Other inequality statements about the h^*-vector are known due to Ehrhart, Stanley, Hibi, and Stapledon among others. We call $h^*(\mathcal{P})$ *unimodal* if there exists a $j \in [d]$ such that $h_i^* \leq h_{i+1}^*$ for all $0 \leq i < j$ and $h_k^* \geq h_{k+1}^*$ for all $j \leq k \leq d$. The cause of unimodality for h^*-vectors in Ehrhart theory is mysterious. Schepers and van Langenhoven [18] have raised the question of whether or not the integer decomposition property alone is sufficient to force unimodality of the h^*-vector for a lattice polytope. A lattice polytope $\mathcal{P}$ has the *integer decomposition property* if, for every positive integer t, and for all lattice points $p \in t\mathcal{P}$, there exists lattice points $p_1, \dots, p_t \in \mathcal{P}$ such that $p = p_1 + \cdots + p_t$. For short, we say $\mathcal{P}$ is IDP.

In general, the interplay of the qualities of a lattice polytope being reflexive, satisfying the integer decomposition property, and having a unimodal h^*-vector is not well-understood [4]. A long standing conjecture of the field is the following.

Conjecture 1.1 (Hibi and Ohsugi [15]). *If $\mathcal{P}$ is a lattice polytope that is reflexive and satisfies the integer decomposition property, then $\mathcal{P}$ has a unimodal Ehrhart h^*-vector.*

When new families of lattice polytopes are introduced, it is of interest to explore how these three properties behave for that family. Further, lattice simplices have been shown to be a rich source of examples and have been the subject of several recent investigations, especially in the context of Conjecture 1.1 [5, 6, 17, 19].

2. Laplacian Simplices

Recently, there is a heightened interest in studying polytopes associated to graphs. Let G be a finite, simple, connected graph. Then G consists of a vertex set $[n] := \{1, \dots, n\}$ and an edge set $\{ij \mid i, j \in [n], i \neq j\}$. Matrices are used to encode all the data from a graph G. For instance, the *unsigned vertex-edge incidence matrix* has rows and columns indexed by the vertices and edges of G, respectively. The matrix has entries a_{ij} such that $a_{ij} = 1$ if vertex i is incident to the edge j and $a_{ij} = 0$ otherwise. One way to associate a polytope to this graph is to interpret the rows of this matrix as the vertices of a polytope. This is called an *edge polytope* in the literature. Many geometric, combinatorial, and algebraic properties of edge polytopes have been established over the past several decades, e.g. [13, 14, 21, 22].

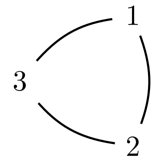
Perhaps the most studied matrix associated to a graph is its *Laplacian matrix*. The Laplacian matrix has rows and columns indexed by $[n]$ and is defined as $L := D - A$ where D is the *degree matrix* and A the *adjacency matrix* of G. Consequently, L has entries $a_{ij} = \deg i$ if $i = j$, $a_{ij} = -1$ if $\{ij\}$ is an edge in G, and $a_{ij} = 0$ else. The first instance the matrix L is explicitly used to associate a polytope to a graph arises in the work of Dall and Pfeifle [8]. They interpret the columns of L as points in $\mathbb{Z}^n$ and form polytopes by considering linear combinations of the line segments with endpoints given by $0 \in \mathbb{Z}^n$ and columns of L. Their analysis revealed interesting polyhedral decompositions and a notable result was an alternative proof of the well-known Matrix Tree Theorem [12]. Other polyhedral associations to the Laplacian matrix include inside-out polytopes studied by M. Beck and B. Braun [3] and Laplacian Eigenpolytopes studied by A. Padrol and J. Pfeifle [16]. My research contribution extends the above list.

With my advisor, I initiated a new way to associate a polytope to a simple graph G using its Laplacian matrix. Consider the rows of the Laplacian matrix as points in $\mathbb{Z}^n$. The convex hull of these points forms a polytope, call it T_G, whose vertices are exactly the rows of L. An example of this construction is provided in Example 2.2. Initially I applied techniques from linear algebra on L to study properties of the polytope T_G, an established combinatorial technique. Since the n rows of L affinely span an $n-1$ dimensional space, T_G is an $n-1$ dimensional simplex. Hence we call T_G the *Laplacian simplex* associated to G. Other notable properties of T_G are the following.

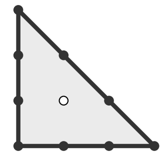
Proposition 2.1 (Braun-M, [7]). *Let G be a connected graph on n vertices.*

(1) T_G contains the origin in its interior.
(2) T_G has normalized volume equal to $n\kappa$, where κ is the number of spanning trees of G.
(3) The h^*-vector of T_G has only positive entries.

Example 2.2. The cyclic graph on 3 vertices, its Laplacian matrix, and its full dimensional associated Laplacian simplex T_G is shown below.



$$L = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$



I explored reflexivity, the integer decomposition property, and unimodality of the Ehrhart h^*-vectors of T_G for special classes of graph. We use C_n to denote the cyclic graph on n vertices and K_n to denote the complete graph on n vertices, that is, K_n has edge set $\{ij \mid \forall i, j \in [n], i \neq j\}$.

Theorem 2.3 (Braun-M, [7])**.** *Let G be a tree, odd cycle, or complete graph, all with $n \geq 3$ vertices. Then Laplacian simplices T_G are reflexive.*

Question 2.1. Which other graphs yield reflexive T_G?

A natural topic to investigate is how operations on a graph translate to polytopal properties of the associated Laplacian simplex. In Prop 3.10 [7], we describe an operation on a graph which preserves the equivalence class of the resulting simplex. This result motivates further research.

Question 2.2. Which polytopes in the equivalence class of T_G can be recognized as Laplacian simplices?

Two graph operations which behave nicely with reflexivity are bridging and whiskering graphs. It is possible to generate a reflexive Laplacian simplex associated to the graph defined by taking two graphs with the same vertex set, each corresponding to a reflexive T_G, and connecting them via the addition of an edge, reminiscent of a bridge (Theorem 3.14 [7]). Another well-studied graph operation is whiskering, which involves attaching a new edge and vertex to each pre-existing vertex in G to obtain the graph $W(G)$. This also has an interesting connection to the reflexivity of T_G.

Theorem 2.4 (Braun-M, [7])**.** *If G is a connected graph on n vertices such that T_G is reflexive or 2-reflexive, then $T_{W(G)}$ is reflexive.*

The property of being 2-*reflexive* is a generalization of reflexive polytopes introduced in [11], in which the polytope contains the origin in its strict interior, its vertices are primitive, and all its facets are integral distance 2 from the origin. Even cycles produce 2-reflexive Laplacian simplices. A notable consequence of Theorem 2.4 is that a whiskered even cycle has a reflexive Laplacian simplex. There are many directions we can take to answer the following question.

Question 2.3. Which other graph operations preserve reflexivity of Laplacian simplices?

We now turn our attention to what can be said about the Ehrhart h^*-vector of these Laplacian simplices. Since T_G is a simplex, $h^*(T_G)$ has a concrete combinatorial interpretation. The entry h_i^* counts the number of lattice points at height i in the *fundamental parallelepiped* of T_G, that is,

$$\Pi_{T_G} := \{\lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_n w_n \mid 0 \leq \lambda_1, \lambda_2, \ldots, \lambda_n < 1\} \subseteq \operatorname{cone}(T_G) \subseteq \mathbb{R}^{d+1}.$$

I used this combinatorial interpretation to prove further results for graph classes C_n and K_n.

Theorem 2.5 (Braun-M, [7])**.** *For $n \geq 2$, the lattice points at height h in $\operatorname{cone}(T_{K_n})$ are in bijection with weak compositions of hn of length n.*

Corollary 2.6 (Braun-M, [7])**.** *The following are consequences of the above theorem.*

1. *The Ehrhart polynomial of T_{K_n} is $L_{T_{K_n}}(t) = \binom{tn+n-1}{n-1}$.*
2. *The h^*-vector of T_{K_n} is $h^*(T_{K_n}) = (1, m_1, \ldots, m_{n-1})$ where m_i is the number of weak compositions of in of length n with parts of size less than n.*

Enumerating points in the fundamental parallelepiped also leads to an explicit description of $h^*(T_G)$ for odd cycles.

Theorem 2.7 (Braun-M, [7]). *Consider C_n where $n \geq 3$ is odd. Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ be the prime factorization of n where $p_1 > p_2 > \cdots > p_k$. Then*

$$h^*(T_{C_n}) = (1, \ldots, 1, h^*_m, h^*_{m+1}, \ldots, h^*_{\frac{n-1}{2}}, \ldots, h^*_{n-m-1}, h^*_{n-m}, 1, \ldots, 1)$$

where $m = \frac{1}{2}(n - p_1^{a_1} \cdots p_k^{a_k - 1})$ and $h_m > 1$.

In particular, odd prime cycles produce a family of reflexive polytopes which satisfy $h^*(T_G) = (1, \ldots, 1, n^2 - n + 1, 1, \ldots, 1)$. This result is exciting as the h^*-vector has an unusual form. From this we are able to show T_G is not IDP whenever G is an odd cycle on $n \geq 5$ vertices.

To study unimodality of $h^*(T_G)$ we rely on the characterization of reflexive polytopes proved by [10]. It states a d-dimensional lattice polytope $\mathcal{P} \subseteq \mathbb{R}^d$ containing the origin in its interior is reflexive if and only if $h^*(\mathcal{P})$ satisfies $h^*_i = h^*_{d-i}$ for $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$.

Theorem 2.8 (Braun-M, [7]). *For odd cycles, $h^*(T_G)$ is unimodal.*

It is also true that whenever G is a tree or a complete graph, $h^*(T_G)$ is unimodal. We can of course continue our quest to find a characterization for unimodal h^*-vectors.

Question 2.4. Which G have unimodal $h^*(T_G)$?

3. Extending Laplacian Simplices

Naturally we inquire what happens to the Laplacian simplices when our graph is no longer simple. This past year I worked to generalize my results with Braun by allowing G to have multiple directed edges. A *directed edge* points from one vertex, called the *tail*, to a different vertex, called the *head*. A *digraph* D is a finite collection of vertices and directed edges in which multiple directed edges between vertices are allowed. Let a_{ij} be the number of directed edges with vertex i as the tail and vertex j as the head where $i, j \in [n]$, the vertex set of D. We do not allow D to have loops; thus $a_{ii} = 0$ for each $i \in [n]$. The number of edges having a tail on vertex i is called the *outdegree* of i and is denoted by $\text{outdeg}(i)$. The *Laplacian matrix* of a digraph D is the $n \times n$ matrix $L(D) := O(D) - A(D)$. Here $O(D)$ is the *outdegree matrix* of D, which contains diagonal entries $\text{outdeg}(i)$ and 0 everywhere else, and $A(D)$ is the *adjacency matrix* of D, which contains entries a_{ij} defined above.

We call D *strongly connected* if it contains a directed path from i to j for every pair of vertices $i, j \in [n]$ and *weakly connected* if there exists a path (not necessarily directed) from i to j for every pair of vertices $i, j \in [n]$. A *converging tree* is a weakly connected digraph having one vertex, called the *root* of the tree, with outdegree zero, while all other vertices have outdegree one. For any $i \in [n]$, we define c_i to be the number of spanning trees which converge to i, i.e. the converging trees of D with n vertices having i as the root. Finally we let $c(D) := \sum_{i=1}^n c_i$ be the total number of converging spanning trees of D. This is usually referred to as the *complexity* of the digraph D.

Through a collaboration with Balletti, Hibi, and Tsuchiya [1], I provided a more general setting to investigate Laplacian simplices. Consider the rows of $L(D)$ as points in $\mathbb{Z}^n$, and consider their convex hull to form the polytope P_D whose vertices are exactly the rows of $L(D)$. We focused on the cases in which a digraph D on n vertices defines an $n-1$ dimensional simplex. This happens precisely when the rank of $L(D)$ is $n-1$, or equivalently, when D has at least one spanning converging tree. In this case, $c(D) \geq 1$, and thus D has positive complexity. This generalizes the Laplacian simplices in Section 2 because the Laplacian of G can be interpreted as the Laplacian of a particular digraph D_G, making the associated simplices equal, that is, $T_G = P_D$. Thus the Laplacian simplices T_G form a subfamily of P_D. The following result generalizes Proposition 2.1.

Proposition 3.1 (Balletti-Hibi-M-Tsuchiya, [1]). *Let D be a digraph on n vertices such that D has positive complexity.*

(1) P_D contains the origin. Further, the origin is in the strict interior of P_D if and only if D is strongly connected.
(2) P_D has normalized volume equal to $c(D)$, the total number of spanning converging trees.
(3) The c_i encode the barycentric coordinates of P_D.

Not only does the family P_D generalize all Laplacian simplices T_G, it also generalizes a reflexive family known as q-simplices [6]. Let $q = (q_1, \dots, q_n)$ be an nondecreasing sequence of positive integers satisfying the condition $q_j|(1 + \sum_{i\neq j} q_i)$ for all $j \in [n]$. For such a vector, the *q-simplex* $\Delta_{(1,q)}$ is defined as

$$\Delta_{(1,q)} := \operatorname{conv}\left\{e_1, e_2, \dots, e_n, -\sum_{i=1}^{n} q_i e_i\right\},$$

where $e_i \in \mathbb{R}^n$ is the i-th standard basis vector. The simplices $\Delta_{(1,q)}$ are a subfamily of Laplacian simplices arising from special star-shaped strongly connected digraphs where we take q_i to be c_i in D.

We use this identification to derive a characterization of reflexive P_D. Let D be a strongly connected digraph such that $\gcd\{c_1, \dots, c_n\} = 1$. Then P_D is reflexive if and only if $c_i|c(D)$ for all i (Corollary 10 [1]). Further, an explicit formula for the h^*-vector as well as a sufficient condition for IDP are given in [6]. We apply these results to those $P_D \cong \Delta_{(1,q)}$.

Question 3.1. Which other families of polytopes can be realized as Laplacian polytopes?

We analyzed a specific class of digraphs, called cycle digraphs. The *cycle digraph* is a set of vertices $[n]$ with directed edges $\{i, i+1\}_{i=1}^{n-1}$, $\{n, i\}$, and $\{j, j-1\}$ for $j \in S \subseteq [n]$. We denote by C_n^S the cycle digraph on n vertices such that $S \subseteq [n]$ is the set of vertices such that $\operatorname{outdeg}(i) = 2$. We characterize reflexivity for Laplacian simplices $P_{C_n^S}$, extending the result of odd cycles in Section 2 (condition (3) below).

Theorem 3.2 (Balletti-Hibi-M-Tsuchiya, [1]). *The Laplacian simplex $P_{C_n^S}$ associated to a cycle digraph C_n^S, is reflexive if and only if one of the following conditions is satisfied:*

(1) $S = \varnothing$, or
(2) $S = [n]$ and $n = 2$, or
(3) $S = [n]$ and n is odd, or
(4) $\varnothing \subsetneq S \subsetneq [n]$, such that $k|c(D)$ for each integer $1 \le k \le K+1$, where K is the longest chain of consecutive edges pointing counterclockwise, i.e.

$$K := \max\{j \mid \{a+1, \dots, a+j\} \subseteq S, \text{ for some } a \in [n]\}.$$

Further characterizations state conditions in which the Laplacian simplex associated to C_n^S contains no other lattice points besides the origin and vertices, as well as conditions in which the Laplacian simplex associated to C_n^S is IDP (Theorem 17 and Theorem 19 [1]). As an interesting application of the tools used to produce the above results, we construct a reflexive Laplacian simplex associated to a cycle digraph having non-unimodal h^*-vector.

Theorem 3.3 (Balletti-Hibi-M-Tsuchiya, [1]). *Let $\alpha, \beta, k \in \mathbb{Z}_{>0}$ such that $\alpha \le \beta \le k-1$ and $\alpha + \beta \le k+1$. Let C_n^S be a cycle digraph, with $n := 6(k+1) - 2\alpha - \beta$, and $S := S_1 \cup S_2 \cup S_3$, with*

$$S_1 := \{1 + 3h \mid 0 \le h \le \alpha - 1\},$$
$$S_2 := \{2 + 3h \mid 0 \le h \le \alpha - 1\},$$
$$S_3 := \{3\alpha + 1 + 2h \mid 0 \le h \le \beta - \alpha - 1\}.$$

Then P_D is a reflexive simplex of dimension $6(k+1)-2\alpha-\beta-1$ with symmetric and nonunimodal h^*-vector

$$(\underbrace{1,\ldots,1}_{2(k+1)-\alpha},\underbrace{2,\ldots,2}_{\alpha},\underbrace{1,\ldots,1}_{(k+1)-\alpha-\beta},\underbrace{2,\ldots,2}_{\beta},\underbrace{1,\ldots,1}_{(k+1)-\alpha-\beta},\underbrace{2,\ldots,2}_{\alpha},\underbrace{1,\ldots,1}_{2(k+1)-\alpha})$$

As seen in Section 2, there are correlations between graph operations on G and properties of associated Laplacian simplex T_G. It would be worthwhile to investigate with D and P_D.

Question 3.2. Are there combinatorial correlations between operations on digraphs and operations on the polytopes P_D?

It is natural to wonder how the structure of the underlying simple graph, denoted $G(D)$, of a digraph D affects the reflexivity of P_D. Here $G(D)$ is the simple graph with vertex set $[n]$ and edge set $E(G) = \{\{i,j\} \subset [n] : a_{i,j} > 0 \text{ or } a_{j,i} > 0\}$.

Question 3.3. For any simple graph G on $[n]$, does there exist a digraph D on $[n]$ such that $G(D) = G$ and P_D is a reflexive $n-1$ dimensional simplex?

We have shown the existence of a simple graph G such that no D with at most one directed edge between each pair of vertices and $G(D) = G$ is a reflexive $n-1$ dimensional simplex. The above question seems unlikely but no counterexample has been found.

4. Undergraduate Involvement

I look forward to sharing my research area with undergraduate students at a future institution. My initial exposure to math research as an undergraduate was through a summer Research Experience for Undergraduates program. Ironically the project was a problem rooted in the field of combinatorics and graph theory, the field in which my current research resides. I was then motivated to spend the following summer doing original research with my advisor to eventually write a thesis. These experiences sparked my desire to pursue a career in mathematics, and I hope to one day inspire my students in an analogous way.

My research area provides ample opportunity to involve undergraduates. The field of combinatorics is notorious for accessible problems to students of varying mathematics exposure. As far as polyhedral combinatorics is concerned, undergraduates are equipped to understand the hyperplane and vertex descriptions of polytopes after a standard linear algebra course. Some results on Laplacian simplices can be obtained through computational techniques from linear algebra. More advanced students with experience in abstract algebra might consider Cayley graphs to generate a new family of polytopes. I am also excited to extend my research interests to incorporate a broader range of problems.

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