Research Statement

Carolyn Marie Yarnall

My current research lies in two different areas of mathematics:

1. The subfield of algebraic topology called equavariant stable homotopy theory.
2. Applications of combinatorial topology and geometry in the social sciences.

1 Equivariant Stable Homotopy Theory

1.1 Background. In algebraic topology, our primary goal is to distinguish topological spaces using algebra. We do this by assigning algebraic objects, often groups, to our spaces in a coherent fashion. The homotopy groups of a space X, denoted $\pi_n(X)$ are a classic example in which we assign groups to a space. $\pi_n(X)$ is formed by taking equivalence classes of maps $S^n \to X$. Homotopy groups give significant information regarding the structure of a space but unfortunately are notoriously difficult to compute. To get around this, we often consider changing spaces into more algebraic objects called spectra on which homotopy calculations are simpler to perform. Stable homotopy theory refers to using spectra.

1.2 Group actions. While homotopy groups provide a significant amount of information about a space or spectrum, we would also like to study the symmetries of a space in this context. We do this in equivariant homotopy theory by letting a group act on a space, or in the stable environment, spectrum X and then use the G-maps $S^n \to X$ to define the homotopy groups. We can encode even more information by considering not only the action of the entire group, but also the action of subgroups of G. The equivariant stable homotopy groups of a spectrum X are computed as $\pi_n(X^H)$, that is, the homotopy groups of the H-fixed points of X for each $H \leq G$. When we keep track of all subgroup data, instead of $\pi_n(X)$ we will write $\underline{\pi_n(X)}$.

1.3 Equivariant Slice Filtration. Often in mathematics, we try to better understand an object by using a filtration to see how it is build from smaller pieces. For example, in algebra, we may look at a nested sequence of normal subgroups to better understand the nature of the entire group. In stable homotopy theory, we also use filtrations to study spectra. The equivariant slice filtration, or slice tower, is a tool that helps us to look at the smaller pieces of a spectrum with a group action.

$$
\begin{array}{ccc}
P^{n+1}_{n+1}X & \longrightarrow & P^{n+1}X \\
 & & \downarrow \\
P^{n}_{n}X & \longrightarrow & P^{n}X \\
 & & \downarrow \\
P^{n-1}_{n-1}X & \longrightarrow & P^{n-1}X \\
 & & \downarrow
\end{array}
$$

To construct the slice tower of a spectrum X with a group action, partially shown at the right, we make a new spectrum P^nX for each layer by making maps from certain spheres of dimension $\geq n$ with group actions trivial. For example, by making the maps $S^n \to X$ trivial, we have $\pi_n(X) = 0$. By doing this we have maps $P^nX \to P^{n-1}X$. We write the fiber, or preimage of a particular point in the image, of such maps as P^n_nX and call it the n-slice of X. The maps $P^nX \to P^{n-1}X$ along with their fibers make up the slice tower of X.

1.4 Current Work & Future Directions. While the slice filtration has been used successfully to prove some interesting results, much about the way it filters objects is unknown. As such, it would be nice to compute the slice tower for a variety of spectra with group actions for different groups. In my dissertation, I computed the slice towers of particular spectra of the form $S^n \wedge H\underline{\mathbb{Z}}$ where the group G was a cyclic p-group for odd primes p. While purely homotopically speaking, such spectra are very simple, when we view them equivariantly there is a lot more interesting structure. I would like to expand my results concerning the spectra described above to cyclic 2-groups. I have already a description pertaining to the cyclic group of order 4. It would also be nice to expand to other cyclic groups whose subgroups are not all nested. This would provide even more information about the slice filtration. Lastly, I would like to expand the scope of my work to include spectra of other forms.

2.1 Applications to Social Science - Voting Theory

2.1.1 Background. In my research, I consider the system of approval voting. In approval voting, voters cast votes for all platforms they approve of rather than voting for only one candidate. The winning candidate is the one whose platform is approved by the most voters.

A *society of voters* $S = (X, V, \mathcal{A})$ consists of X, a spectrum of platforms; V, a set of voters; $\mathcal{A}$, a collection of approval sets A_v for each $v \in V$. For example, we might consider the society of voters in the U.S. voting for presidential candidates. The spectrum could be thought of as $\mathbb{R}$ where the further left we look, the more liberal the platform, and the further right, the more conservative. Then the approval sets in $\mathcal{A}$ would be intervals on this line reflecting each voters political views. Such a society is called a *linear society.* For any society, the size of the society, written $|S|$, is the number of voters.

The *agreement number* $a(S)$ of a society is the maximum number of voters that approve the same platform. Of primary concern is the question: For a given society S, can we determine a lower bound on $a(S)$? For instance, is is possible to know that there will be a platform that is approved by at least half the voters? To attempt to answer this question, it can be helpful to identify the agreeability of the society. A (k, m)-*agreeable* society is one in which among every collection of m voters, at least k of them agree on a platform.

2.1.2 Product Societies. I am currently studying (along with K. Mazur, M. Sondjaja, and M. Wright) voting societies under approval voting that may be considered as a product of other societies. For example, we may consider a society with a spectrum $X = \mathbb{R} \times \mathbb{R}$ where the two linear societies may reflect two different spectra of platform types. An approval set in this society is a product of two intervals, or a rectangle in the plane. In this context, we ask: If we know the agreement number of particular societies, what can we say about the agreement number of their product?

We have the following result: **Theorem** Suppose $S = S_1 \times S_2$ is a (k, m)-agreeable society. If S is large enough, $a(S) \geq a(S_1)a(S_2)$.

We are currently working on analyzing this bound to see how good it is. For example, when looking at the society with $X = \mathbb{R} \times \mathbb{R}$ described above, there are other bounds already known for the agreement number. We would like to determine, if possible, an example for which our bound is strict or at least smaller than the known bound.

2.1.3 Helly's Theorem. For (k, m)-agreeable linear societies, there are already determined bounds for the agreement number. When $k = m = 2$, sometimes called a *super-agreeable* society, the agreement number is the size of the society. In other words, there is a platform that is approved by all voters. The proof follows from Helly's Theorem: Given n convex sets in $\mathbb{R}^d$ where $n > d$, if every $d+1$ of them intersect at a common point, then they all intersect at a common point. Of course, for linear societies we only need the case when $d = 1$. The proof is enlightening in that it givens an idea of the position of the point of intersect relative to the sets.

When $m > k$ there is also a result about the agreement number. However, the proof relies on translating the society using graph theory and this does not give much information about the location of the platform agreed upon by the most candidates. I have begun working (along with K. Mazur, A. Ruiz, and F. Su) on developing a proof akin to the proof of Helly's Theorem. This would give a better idea about what (k, m)-agreeable linear societies must look like and where the maximum number of voters in agreement is likely to occur.

2.2 Fair Division Problems

In dividing up a collection of goods among a group of people, if we allow each person to express preferences over the various options, we might ask: Is there a way to divide the goods in such a way that no person prefers another's allotment more than their own? This is the question we aim to answer when looking for an *envy free* division of goods. As the collection of options can often be thought of as a topological or geometric space, we can use results from such fields to aid in the search for an envy-free division. One such result is **Sperner's Lemma**: Any Sperner-labelled triangulation of an n-simplex must contain an odd number of fully labelled elementary n-simplices. In particular, there is at least one.

Sperner's Lemma can be applied to fair division problems in such a way that locating a fully labelled simplex corresponds to locating an envy-free division. In a paper by F. Su, we see that since there is a constructive proof of Sperner's Lemma, not only can we guarantee in many cases that there will be an envy-free division, but there is an algorithm that will locate such a division. This method works primarily for (reasonably) continuously divisible goods, though it can be adapted to indivisible or discrete objects in certain cases.

There is still much to consider in this area. For example, the algorithm mentioned above works only in the case we have one collection to divide. The case where we have more than one collection or type of object to portion out is much more difficult and in some cases, there is no envy-free division possible. I would like to continue working on some variants of these fair division problems.

2.3 Undergraduate Research

In either of the areas mentioned in 2.1 and 2.2, undergraduates could easily be involved. There are a collection of papers relating to these areas that are readable and I expect that not only could I involve students in the work I am currently performing, but they could form their own questions about this work that we could investigate together. Please see Francis Su's letter regarding the possibilities in working with students in this area.