(UNSTABLE) $\mathbb{A}^1$-HOMOTOPY THEORY

BERTRAND GUILLOU

1. Introduction

The goal will be to "construct" a homotopy theory in the context of smooth schemes over a given base field k in which the affine line $\mathbb{A}^1_k$ plays the role of the unit interval.

2. Crash Course in Algebraic Geometry

The following "dictionary" will give us the necessary intuition:

Algebraic Geometry	Topology
$\mathbb{A}^n_k$	$\mathbb{R}^n$
Affine Scheme	Variety in $\mathbb{R}^n$
Scheme	Manifold
Separated Scheme	Hausdorff
Proper Scheme	Compact
Smooth scheme	Smooth Manifold
Smooth morphism of rel. dim. n	Submersion of codim n
étale morphism (smooth of rel. dim. 0)	Local homeomorphism

3. Foundations

By now we have been sufficiently brainwashed so that when we hear "homotopy theory" we think of model categories. These days, a model category is usually required to be complete and cocomplete, so that one can do things like the small abject argument in order to build things functorially. Of course, Sm_k is not complete and cocomplete. To fix this, we just consider instead the category of formal colimits in Sm_k. To be more precise, recall

Definition 1. A *presheaf* $\mathcal{F}$ on a category $\mathscr{C}$ is simply a contravariant functor $\mathcal{F} : \mathscr{C}^{op} \to Set$. We will denote the category of presheaves on $\mathscr{C}$ by $Pre(\mathscr{C})$.

Example 1. Given any $X \in \mathscr{C}$, we have the presheaf h_X defined by

$$h_X(Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X).$$

We will also need the following trivial, but fundamental, fact:

Proposition 1 (Yoneda's Lemma). *Given any presheaf $\mathcal{F}$ on $\mathscr{C}$, there is a natural isomorphism*

$$\mathrm{Hom}_{Pre(\mathscr{C})}(h_X, \mathcal{F}) = \mathrm{Nat}(h_X, \mathcal{F}) \cong \mathcal{F}(X).$$

In particular, $\mathrm{Hom}_{Pre(\mathscr{C})}(h_X, h_Y) \cong \mathrm{Hom}_{\mathscr{C}}(X, Y)$.

So now we may embed Sm_k into the category $Pre(\mathrm{Sm}_k)$ of presheaves on Sm_k via the Yoneda embedding h. By abuse of notation, if $X \in \mathrm{Sm}_k$, we will also write X for the corresponding represented presheaf on Sm_k.

The category $Pre(\mathrm{Sm}_k)$ is complete and cocomplete (limits and colimits are simply computed objectwise). In fact, it is a universal cocompletion of Sm_k, in the following sense:

Proposition 2. *If $F : \mathrm{Sm}_k \to \mathscr{D}$ is any functor to a cocomplete category $\mathscr{D}$, then there is a* unique *colimit-preserving functor $\tilde{F} : Pre(\mathrm{Sm}_k) \to \mathscr{D}$ such that the diagram*

$$\begin{tikzcd} \mathrm{Sm}_k \arrow[r, "h"] \arrow[dr, "F"'] & Pre(\mathrm{Sm}_k) \arrow[d, dashed, "\exists! \tilde{F}"] \\ & \mathscr{D} \end{tikzcd}$$

commutes. Moreover, $\tilde{F}$ has a right adjoint $G : \mathscr{D} \to Pre(\mathrm{Sm}_k)$ given by

$$G(d)(X) = \mathrm{Hom}_{\mathscr{D}}(F(X), d).$$

The key fact here is that every presheaf is canonically isomorphic to a colimit of representable presheaves. Since $\tilde{F}$ is already determined by F on the representable presheaves, there is no choice left for defining $\tilde{F}$ on an arbitrary presheaf. Again, one should think of the Yoneda embedding as the process of formally adjoining colimits to a given category.

At this point, we feel pretty good about embedding Sm_k inside $Pre(\mathrm{Sm}_k)$. Nevertheless, we run into a problem. Namely, suppose that we have a Zariski cover $X = U \cup V$. In other words, we have a pushout diagram

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & X \end{tikzcd}$$

in Sm_k. Unfortunately, the corresponding diagram in $Pre(\mathrm{Sm}_k)$ is no longer a pushout. Indeed, the identity map of X, considered as an element of $X(X)$, does not show up in the pushout $U(X) \cup_{U \cap V(X)} V(X)$. Morally, we have formally adjoined colimits, but we already had some colimits to begin with, and we want to define the old colimits with ones in our new category. The solution of this problem is the passage to sheaves.

The idea is the following: we want, for example, the canonical map

$$U \cup_{U \cap V} V \to X \tag{1}$$

in $Pre(\mathrm{Sm}_k)$ to be an isomorphism. We can consider it as an isomorphism by restricting our attention to those presheaves which believe this map to be an isomorphism[1]. A Zariski sheaf is exactly a presheaf which believes every natural transformation of the form (1) to be an isomorphism.

In fact, Zariski covers won't quite work for us. Moreover, we would like to talk about sheaves on an arbitrary category, and so we will have to specify what we mean by a cover in a category.

Definition 2. A *Grothendieck (pre-)topology* on a category $\mathscr{C}$ with fiber products is, for each object $X \in \mathscr{C}$, a collection of families $\{U_\alpha \to X\}$ of morphisms, called covering families, which satisfy the following properties:

(i) any isomorphism constitutes a covering family

(ii) if $\{U_\alpha \to X\}$ is a covering family and $Y \xrightarrow{f} X$ is any morphism, then $\{U_\alpha \times_X Y \to Y\}$ is also a covering family

[1] This process should be familiar from our earlier discussions of localization of a category with respect to a set of maps. Sheaves are merely the local objects with respect to the set of covering morphisms.

(iii) if $\{U_\alpha \to X\}$ is a cover and, for each α, $\{V_{\alpha\beta} \to U_\alpha\}$ is a cover, then $\{V_{\alpha\beta} \to X\}$ is a cover.

Example 2. In the Zariski topology on Sm_k, a cover $\{U_\alpha \to X\}$ is given by a finite, surjective family of open immersions.

Example 3. In the Nisnevich topology on Sm_k, a cover $\{U_\alpha \to X\}$ is given by a finite, surjective family of étale morphisms such that for each $x \in X$, there exists a $u \in U_\alpha$ such that $k(x) \xrightarrow{\cong} k(u)$.

Definition 3. If $\mathscr{C}$ is a category with a given Grothendieck pre-topology, we say that a presheaf $\mathcal{F}$ on $\mathscr{C}$ is a *sheaf* (with respect to the given topology), if for every $X \in \mathscr{C}$ and every cover $\{U_i \to X\}_{i\in I}$ of X, the canonical map

$$\mathcal{F}(X) \longrightarrow \mathrm{eq}\left(\prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(U_i \times_X U_j)\right)$$

is an isomorphism.

In particular, we will be interested in Nisnevich sheaves. It turns out that there is a particularly nice characterization of Nisnevich sheaves. First we need a definition.

Definition 4. A pullback square

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & V \arrow[d, "p"] \\ U \arrow[r, "i"] & X \end{tikzcd}$$

is said to be *elementary distinguished* if i is an open immersion and p is étale and induces an isomorphism

$$p^{-1}(X-U)_{red} \xrightarrow{\cong} (X-U)_{red}$$

of closed subschemes.

Proposition 3. *A presheaf $\mathcal{F}$ is a Nisnevich sheaf if and only if it takes elementary distinguished squares to pullback squares of sets.*

Corollary 1. *Every representable presheaf is a Nisnevich sheaf.*

Proof. Indeed, an elementary distinguished square is a pushout square in Sm_k. ■

Corollary 2. *Every elementary distinguished square is a pushout square of Nisnevich sheaves.*

3.1. Building in the simplicial structure

Now we have our category of Nisnevich sheaves, which is complete and cocomplete, but we are still not ready to do homotopy theory. As above, we will "cheat" by embedding our category of sheaves inside a category which has an obvious homotopy theory. Namely, consider

$$d : Shv_{Nis}(\mathrm{Sm}_k) \hookrightarrow sShv_{Nis}(\mathrm{Sm}_k);$$

that is, we send a sheaf $\mathcal{F}$ to the discrete simplicial sheaf $\mathcal{F}$. One can then put a model structure on the category of simplicial sheaves; in fact there are many choices.

It will be more convenient for us to work at the level of presheaves. Note that we can either regard a simplicial presheaf as a presheaf of simplicial sets or as a simplicial object in the category of presheaves (of sets). There are in fact several model structures on the category of simplicial presheaves. We will work with the projective, or Bousfield-Kan model structure.

The weak equivalences and fibrations in this model structure are just the objectwise weak equivalences and fibrations. In other words, a map $\mathcal{F} \to \mathcal{G}$ of simplicial presheaves is a weak equivalence (or fibration) if and only if $\mathcal{F}(U) \to \mathcal{G}(U)$ is a weak equivalence (or fibration) of simplicial sets for all U. This determines the cofibrations to be those with the appropriate lifting property. This model category enjoys a similar universal property to that discussed above for $Pre(\mathrm{Sm}_k)$:

Proposition 4 (Dugger)**.** *Given a functor $F : \mathrm{Sm}_k \to \mathscr{M}$ to a model category $\mathscr{M}$, there is a left Quillen functor $\tilde{F} : sPre(\mathrm{Sm}_k) \to \mathscr{M}$ and a natural weak equivalence $\eta : \tilde{F} \circ (c \circ h) \simeq F$:*

$$\begin{tikzcd} \mathrm{Sm}_k \arrow[r, "c\circ h"] \arrow[dr, "F"'] & sPre(\mathrm{Sm}_k) \arrow[d, dashed, "\exists \tilde{F}"] \\ & \mathscr{M} \end{tikzcd}$$

commutes. Moreover, the category of extensions $\tilde{F}$ is contractible (in a sense made more precise by Dugger).

At least if $\mathscr{M}$ is simplicial, it is not difficult to describe what to do: we know how we have to define $\tilde{F}$ on simplicially constant presheaves. We then extend to an arbitrary simplicial presheaf by use of a coend:

$$\tilde{F}(\mathcal{G}) = (F \otimes \Delta^\bullet) \otimes_{\mathrm{Sm}_k} \mathcal{G}.$$

One should regard the process of passage from Sm_k to $sPre(\mathrm{Sm}_k)$ as formally adjoining all homotopy colimits.

As above, we may have diagrams in Sm_k which we wanted to regard as colimits, and we now want to identify these with appropriate homotopy colimits. In particular, given any Nisnevich cover $\mathcal{U} = \{U_i \to X\}_{i \in I}$, we can form the associated Čech simplicial presheaf $\mathcal{U}_\bullet$, given in dimension n by

$$\mathcal{U}_n = \coprod_{i_0,\dots,i_n \in I} U_{i_0} \times_X U_{i_1} \times_X \cdots \times_X U_{i_n}.$$

We then perform Bousfield localization at the maps $\mathrm{hocolim}\, \mathcal{U}_\bullet \to X$. In fact, since Nisnevich sheaves are detected on the elementary distinguished squares, it suffices to localize at those particular covers. Let us denote our resulting homotopy category by $\mathcal{H}o(k)$.

4. The $\mathbb{A}^1$-homotopy category

As we said at the beginning, we want the affine line $\mathbb{A}^1_k$ to play the role of the unit interval in our homotopy theory. In particular, the projection $X \times \mathbb{A}^1_k \to X$ should be a weak equivalence. But of course we know how to force this to be true: we can just localize at all such projections. The $\mathbb{A}^1$ model structure on simplicial presheaves is the Bousfield localization of our previous model structure at the maps $X \times \mathbb{A}^1 \to X$. The resulting homotopy category will be denoted by $\mathcal{H}o_{\mathbb{A}^1}(k)$.

Note that by construction $X \times \mathbb{A}^1_k$ is a cylinder object for X in this model structure.

As in the classical case, one often wants to work with pointed spaces. A pointed simplicial presheaf is simply a simplicial presheaf $\mathcal{F}$ together with a morphism $* \to \mathcal{F}$, where $*$ is the discrete simplicial presheaf represented by $* = \mathrm{Spec}(k)$.

We now inherit a model structure on pointed simplicial presheaves by declaring a map of pointed simplicial presheaves to be a weak equivalence, fibration, or cofibration if it already is when forgetting about basepoints. We denote the associated homotopy category by $\mathcal{H}o_{\mathbb{A}^1,\bullet}(k)$.

5. Constructions

Many constructions from classical homotopy theory carry over in this situation.

For $n \geq 0$, define

$$\Delta^n = \mathrm{Spec}(k[t_0, \ldots, t_n]/(\sum_i t_i = 1)) \subset \mathbb{A}_k^{n+1}.$$

Just as in the classical case, as n varies this yields a cosimplicial object in Sm_k. This allows us to define a "chains" functor $C_\bullet : Pre(\mathrm{Sm}_k) \to sPre(\mathrm{Sm}_k)$ by $C_\bullet(\mathcal{F}) = \mathrm{Hom}(\Delta^\bullet, \mathcal{F})$. Recall that the internal hom of presheaves is defined as

$$\mathrm{Hom}(\Delta^\bullet, \mathcal{F})(U) = \mathcal{F}(U \times \Delta^\bullet) = \mathrm{Hom}(U \times \Delta^\bullet, \mathcal{F}).$$

Now we can use our cosimplicial scheme $\Delta^\bullet$ to build a left adjoint to $C_\bullet$. Namely, the left adjoint is geometric realization, and it is defined, like in the classical case, as a coend

$$|\mathcal{F}_\bullet| = \Delta^\bullet \otimes_\Delta \mathcal{F}_\bullet.$$

One can in fact use this adjoint pair to transport the model structure on $sPre(\mathrm{Sm}_k)$ to one on $Pre(\mathrm{Sm}_k)$.

Moreover, one can show that if $\mathcal{F}$ is a simplicial presheaf then $\mathcal{F} \simeq |\mathcal{F}|$, where the geometric realization is considered as a discrete simplicial presheaf. In particular, $|\Delta^n| \simeq \mathbb{A}^n$.

There are a couple of candidates for "spheres" in our homotopy category. First, the simplicial circle $S_s^1 = \Delta^1/\partial\Delta^1$, considered as constant in the presheaf direction, is an obvious choice. Second, at least if we are working over $\mathbb{C}$, then $\mathbb{A}^1 - 0 = \mathrm{Spec}(k[t, t^{-1}])$ should at least have the homotopy type of a circle; it's the best we can do for a general field k. This scheme is also known by the name of $\mathbb{G}_m$, as it is a group scheme under multiplication. We will see another candidate soon.

Given two pointed (simplicial) presheaves (X, x) and (Y, y), we define their smash product to be

$$X \wedge Y = X \times Y/((X \times \{y\}) \cup (\{x\} \times Y)).$$

Given a vector bundle $E \to X$, we define the Thom space $Th(E)$ to be the quotient presheaf $E/E - s(X)$, where $s : X \to E$ is the zero section. We emphasize that this is a quotient of presheaves. We define the Tate object T to be the Thom space of the trivial line bundle over a point:

$$T = Th(\mathbb{A}^1 \to pt) = \mathbb{A}^1/(\mathbb{A}^1 - 0).$$

Note that, in topology, the Thom space of a trivial line bundle gives a circle. In fact, our three candidate circles are related in the following way:

Proposition 5. *$S_s^1 \wedge \mathbb{G}_m \simeq T \simeq \mathbb{P}^1$.*

Proof. Let X be the pushout

$$\begin{tikzcd} \mathbb{G}_m \arrow[r] \arrow[d, "1\times \mathrm{id}"'] & \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 \wedge \mathbb{G}_m \arrow[r] & X \end{tikzcd}$$

Since $\mathbb{A}^1 \wedge \mathbb{G}_m \simeq *$, we get a weak equivalence $X \simeq T$. Similarly, since $\mathbb{A}^1 \simeq *$, we get a weak equivalence $X \simeq \mathbb{A}^1 \wedge \mathbb{G}_m/\mathbb{G}_m = \mathbb{A}^1/\{0,1\} \wedge \mathbb{G}_m \simeq S_s^1 \wedge \mathbb{G}_m$. We have here used the fact that any simplicial presheaf is weakly equivalent to its geometric realization (discussed below).

Also, note that open excision gives

$$T = \mathbb{A}^1/(\mathbb{A}^1 - 0) \cong \mathbb{P}^1/(\mathbb{P}^1 - 0) \simeq \mathbb{P}^1/* \cong \mathbb{P}^1.$$

■

Remark 1. Over the reals, $\mathbb{P}^1$ gives a circle; with the above proposition this supports T's claim to being a circle. On the other hand, over $\mathbb{C}$, $\mathbb{P}^1$ gives a 2-sphere, which supports the claim that S^1_s and $\mathbb{G}_m$ are circles.

When working with Thom spaces it is in fact more convenient to sheafify, as the next proposition shows (the proposition fails before sheafification)

Proposition 6. *$Th(E \times E') \cong Th(E) \wedge Th(E')$ (as sheaves).*

In particular, when E' is a trivial line bundle, we get $Th(E \times \mathcal{O}) \cong Th(E) \wedge T$. This of course implies $Th(E \times \mathcal{O}^n) \cong Th(E) \wedge T^n$. Taking E to be the zero bundle gives in particular

$$\mathbb{A}^n/(\mathbb{A}^n - 0) \cong T^n.$$

The following theorem is one place where the use of a topology at least as strong as the Nisnevich topology is required.

Theorem 1. *(Homotopy Purity) Let $i : Z \hookrightarrow X$ be a closed immersion of smooth schemes, with normal bundle $N_X Z$. Then $Th(N_X Z)$ is $\mathbb{A}^1$-weak equivalent to $X/(X - Z)$.*