The Adams Spectral Sequence

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Introduction

The Adams spectral sequence is one of the most important tools in stable homotopy theory. It allows one to pass from homological information to "homotopical" information (whatever that means). In principle, it somehow streamlines and enhances the Serre spectral sequence computations of homotopy groups of spheres. Unlike Serre spectral sequence, Adams spectral sequence deals with stable homotopy groups only. To emphasize this, I will commit to working with spectra throughout the lecture. In addition to this, I will expand to define the Adams spectral sequence for some generalized cohomology theory, E, where E, of course, refers to both the cohomology theory and the spectrum representing it. There are hypotheses that we are going to impose on our cohomology theories, which may seem a bit restrictive. However, they are not as bad, in light of the fact that the theories of interest to us mostly adhere to them (maybe after some modifications).

I will mainly be concerned by the construction of the spectral sequence. Therefore, let the scarcity (or maybe complete lack) of examples not discourage the audience. The approach is that of Haynes Miller's in [**HRM**]. It is referenced in [**COCTALOS**], which I am going to shamelessly copy here. The nice thing about the approach is the fact that it mimics the homological algebra, that we all have learned to love. Without further ado (I've already spent enough time with this introduction), let us begin.

1. Definitions

I will assume that people know about spectra, their relation to (co)homology theories, and smash products of spectra. Most of the information is in [**ABB**].

Definition 1.1. *(i) A sequence of spectra $A_1 \longrightarrow A_2 \longrightarrow \dots \longrightarrow A_n$ is exact if the sequence of homotopy functors it represents is exact.*

(ii) A map $A \longrightarrow B$ is a monomorphism if $* \longrightarrow A \longrightarrow B$ is exact.

(iii) A map $A \longrightarrow B$ is a epimorphism if $A \longrightarrow B \longrightarrow *$ is exact.

(iv) A sequence $A \longrightarrow B \longrightarrow C$ is short exact if $* \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow *$ is exact.

Remark 1.2. Here the homotopy functor that represents A is the functor $[A, -]$ (and not $[-, A]$). In particular, we can conclude that any cofiber sequence is exact. These two notions in some sense are very closely related.

The monomorphisms and epimorphisms end up being actually quite simple.

Lemma 1.3. *If $f : A \longrightarrow B$ is a mono, then there is a map $g : C \longrightarrow B$, such that $f \vee g : A \vee C \longrightarrow B$ is a weak equivalence. If $g : A \longrightarrow B$ is epi, then there is a homotopy section $r : B \longrightarrow A$, i.e. $gr \simeq \mathbb{1}$, and a map $f : F \longrightarrow A$, such that $r \vee f : B \vee F \longrightarrow A$ is a weak equivalence.*

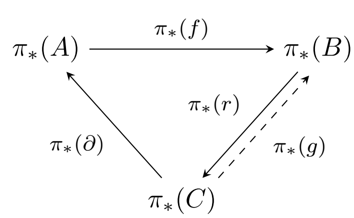
Proof. Note that $[\Sigma E, -] \cong [E, \Sigma^{-1} -]$, which implies that Σ preserves the exactness of sequences of spectra. Then we look at the following cofiber sequence of spectra

$$A \xrightarrow{f} B \xrightarrow{r} C \xrightarrow{\partial} \Sigma A \xrightarrow{-\Sigma f} \Sigma B$$

Note that $-\Sigma f$ is mono. Then we have an exact sequence

$$[C, \Sigma A] \xleftarrow{\partial_*} [\Sigma A, \Sigma A] \xleftarrow{(-\Sigma f)_*} [\Sigma B, \Sigma A]$$

$(-\Sigma f)_*$ is surjective, forcing $\partial_* = 0$. Thus, $\partial \simeq *$. Then recall that the sequence $B \xrightarrow{r} C \xrightarrow{*} \Sigma A$ is a fiber sequence. Then $r : [C, B] \longrightarrow [C, C]$ is surjective. Pick a lift for the identity and call it $g : C \longrightarrow B$. This is a homotopy section of r. Then we look at the exact triangle:



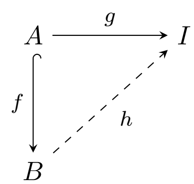
Note that $\pi_*(\partial) = 0$ and $\pi_*(r)$ admits a section (namely, $\pi_*(g)$), and we observe that we have a split short exact sequence. Thus, $\pi_*(f \vee r) = \pi_*(f) \oplus \pi_*(r)$ is an iso.

The statement about epis can be derived from the first part, by taking the fiber $f : F \longrightarrow A$ of $g : A \longrightarrow B$ and then observing that f is mono. ■

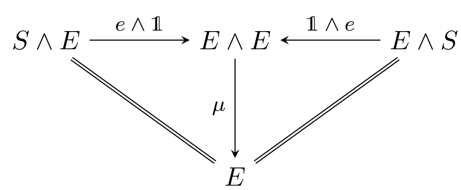
Observe that the splitting is not natural, since it involved a choice of a lift.

Definition 1.4. *A sequence of spectra is E-exact if the sequence becomes exact after smashing with E. The rest of the (E-)notions from 1.1 are defined similarly.*

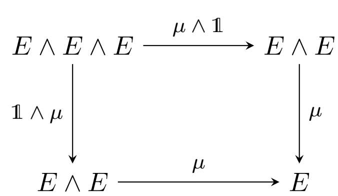
Definition 1.5. *A spectrum I is E-injective if for each E-mono $f : A \longrightarrow B$, and each map $g : A \longrightarrow I$, there is an up to homotopy solution to the diagram*



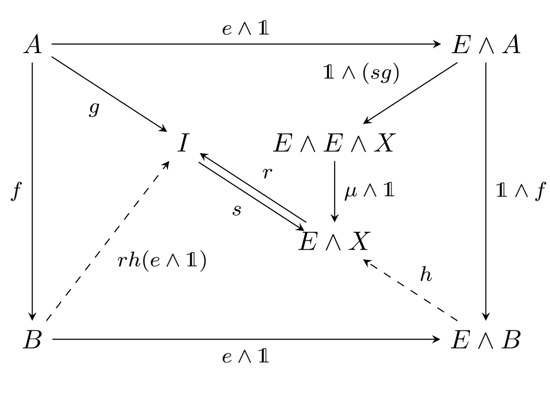
Sidenote 1.6. Miller's approach in defining these notions is different. Fortunately, one can reconcile some of the differences. Miller considers ring spectra right away, so let us do the same. From this point on we will assume that E is a ring spectrum, with structure maps $e : S \longrightarrow E$ (unit) and $\mu : E \wedge E \longrightarrow E$, such that the following diagram commutes up to homotopy



We may also impose the condition that E is homotopy associative, though I don't think it is strictly necessary to prove some of the statements that I am about to give. Homotopy associativity means that the following diagram is homotopy commutative

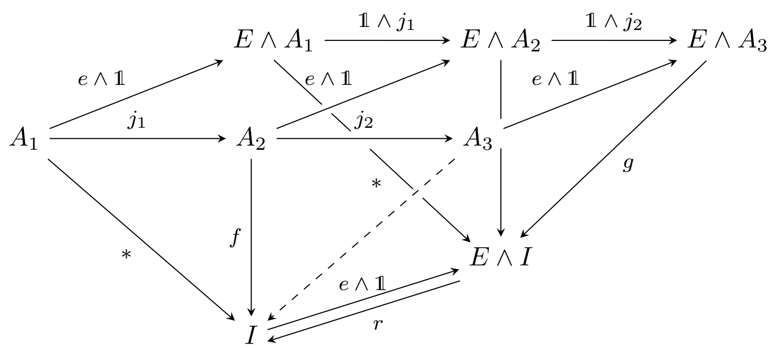


Miller calls a spectrum I E-injective if it is a retract of $E \wedge X$ for some spectrum X. Our notion of E-injectivity implies this. For any spectrum X, the map $e \wedge \mathbb{1} : X \longrightarrow E \wedge X$ is E-mono. Indeed, $\mathbb{1} \wedge e \wedge \mathbb{1} : E \wedge X \longrightarrow E \wedge E \wedge X$ admits a homotopy retraction, namely, the map $\mu \wedge \mathbb{1}$. Then, by E-injectivity of I, we obtain a homotopy retraction $r : E \wedge I \longrightarrow I$. The implication goes the other way too. To see this consider the diagram



The map h exists, since $E \wedge A \longrightarrow E \wedge B$ is monic. Just to check: $rh(e \wedge \mathbb{1})f = rh(\mathbb{1} \wedge f)(e \wedge \mathbb{1}) = r(\mu \wedge \mathbb{1})(\mathbb{1} \wedge (sg))(e \wedge \mathbb{1}) = r(\mu \wedge \mathbb{1})(e \wedge (sg)) = rsg = g$.

Another thing that may be somewhat unclear is Miller's definition of E-exactness. By his definition, $A_1 \longrightarrow A_2 \longrightarrow \ldots \longrightarrow A_n$ is E-exact if we get an exact sequence when we apply the functor $[-, I]$ for E-injective spectra I. Our definition implies that of Miller. Just consider the diagram:



Since, $E \wedge A_1 \longrightarrow E \wedge A_2 \longrightarrow E \wedge A_3$ is exact, we get the map g. Then we consider, $rg(e \wedge \mathbb{1}) : A_3 \longrightarrow I$. It is a solution to our diagram: $rg(e \wedge \mathbb{1})j_2 = rg(\mathbb{1} \wedge j_2)(e \wedge \mathbb{1}) = r(\mathbb{1} \wedge f)(e \wedge \mathbb{1}) = r(e \wedge \mathbb{1})f = f$. I am not sure whether the implication goes in the other direction or not. At this point we know that our assumptions provide us with a more general setting. I've spent enough time with this sidenote. Let's go.

When people define injective objects after that they usually define resolutions.

Definition 1.7. *An E-Adams resolution (or an E-resolution) of a spectrum X is an E-exact sequence*

$$* \longrightarrow X \xrightarrow{i_0} I_0 \xrightarrow{i_1} I_1 \xrightarrow{i_2} \cdots$$

such that I_j's are E-injective and $i_{s+1}i_s \simeq *$ for all $s \geq 0$.

Remark 1.8. In general, the fact that we have an E-exact sequence does not imply $i_{s+1}i_s \simeq 0$. Indeed, suppose we have the obvious map $H\mathbb{Z} \longrightarrow H\mathbb{Z}_{/2}$. ($HG$ is the Eilenberg-MacLane spectrum for the abelian group G.) Smashing it with $H\mathbb{Q}$ produces $H\mathbb{Q} \wedge H\mathbb{Z} \longrightarrow H\mathbb{Q} \wedge H\mathbb{Z}_{/2}$. The target spectrum is trivial, since $\pi_*(H\mathbb{Q} \wedge H\mathbb{Z}_{/2}) = H_*(H\mathbb{Z}_{/2}; \mathbb{Q}) = 0$. However, in our setting, where the target maps are E-injective the statement does follow through. Namely, suppose I is E-injective, and we are given a map $\nu : A \longrightarrow I$, such that $\mathbb{1} \wedge \nu : E \wedge A \longrightarrow E \wedge I$ is null. If $\sigma : I \longrightarrow B$ is the cofiber of ν, then $\mathbb{1} \wedge \sigma : E \wedge I \longrightarrow E \wedge B$ is the cofiber of $\mathbb{1} \wedge \nu \simeq *$. This easily implies that $\mathbb{1} \wedge \sigma$ is mono, or that σ is E-mono. Then we get a retract $r : B \longrightarrow I$. Thus, $\nu = r\sigma\nu \simeq *$. We conclude that the condition $i_{s+1}i_s \simeq *$ is redundant.

Whenever people define resolutions after that what they want to check is whether or not they behave like a derived category.

Proposition 1.9. *Given a diagram of form*

$$\begin{array}{ccccccccc}
* & \longrightarrow & X & \xrightarrow{i_0} & I_0 & \xrightarrow{i_1} & I_1 & \xrightarrow{i_2} & \cdots \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \\
* & \longrightarrow & Y & \xrightarrow[j_0]{} & J_0 & \xrightarrow[j_1]{} & J_1 & \xrightarrow[j_2]{} & \cdots
\end{array}$$

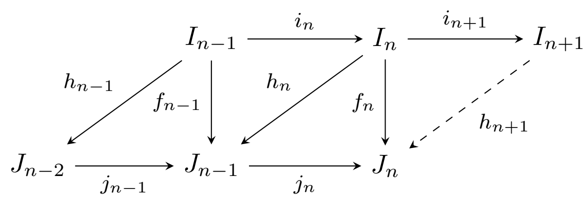
where the horizontal sequences are E-exact, and f is any map, there is a lift of f to a map of resolution. Furthermore, this lift is unique up to chain homotopy.

Proof. Suppose we have lifted the map up to n-th level (regard f at level -1). Then we have a diagram

$$\begin{array}{ccccc}
I_{n-1} & \xrightarrow{i_n} & I_n & \xrightarrow{i_{n+1}} & I_{n+1} \\
\downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} \\
J_{n-1} & \xrightarrow[j_n]{} & J_n & \xrightarrow[j_{n+1}]{} & J_{n+1}
\end{array}$$

Note that $j_{n+1}f_n i_n = j_{n+1}j_n f_{n-1} \simeq *$. Then according to 1.6, there exists f_{n+1}, such that $f_{n+1}i_{n+1} = j_{n+1}f_n$.

For the second part of the proposition, it suffices to prove that there is contracting chain homotopy for $f \simeq *$. Set $h_0 = *$. Then suppose we have constructed the chain homotopy h up to level n. We have the following diagram,



We know that $f_{n-1} = h_n i_n + j_{n-1}h_{n-1}$. Then we observe that $(f_n - j_n h_n)i_n = f_n i_n - j_n h_n i_n = j_n f_{n-1} - j_n f_{n-1} + j_n j_{n-1} h_{n-1} = *$. This implies that there exist h_{n+1}, such that $f_n = h_{n+1}i_{n+1} + j_n h_n$. ■

Another thing people would demand is the existence of resolutions. Let's show that there are E-resolutions. Here, we will definitely need that fact that E is an associative ring spectrum. The following resolution is called *canonical* or *standard*.

Lemma 1.10. *Let $I_n = E^{\wedge(n+1)} \wedge X$ and let $\delta^i : I_n \longrightarrow I_{n+1}$ be the map $\mathbb{1}^{\wedge i} \wedge e \wedge \mathbb{1}^{\wedge(n+1-i)} \wedge \mathbb{1}_X$, for $i \in \{0, 1, \dots, n+1\}$. Then the sequence*

$$* \longrightarrow X \xrightarrow{\ \delta\ } I_0 \xrightarrow{\ \delta\ } I_1 \xrightarrow{\ \delta\ } \cdots$$

where $\delta : I_n \longrightarrow I_{n+1}$ is the map $\sum_{i=0}^{n+1}(-1)^i\delta^i$, is an E-resolution.

Proof. It is fairly clear that I_n's are injective. Thus, we need to show that the sequence is E-exact. In fact what we are dealing with here is a cosimplicial spectrum, i.e. functor $\mathbf{\Delta} \longrightarrow \mathbf{Spec}$, where $\mathbf{\Delta}$ is the category of simplicial objects and **Spec** is the category of spectra. This cosimplicial spectrum sends $[n]$ to $E \wedge I_{n-1}$ (regard, $X = I_{-1}$). The coface maps are $d^i = \mathbb{1} \wedge \delta^i$, and codegeneracy maps are $s^j = \mathbb{1}^{\wedge j-1} \wedge \mu \wedge \mathbb{1}^{\wedge(n-j+1)} \wedge \mathbb{1}_X : I_{n+1} \longrightarrow I_n$. It is rather tedious to check the cosimplicial identities, so we'll skip doing that. Here are the identities,

$$\begin{aligned} d^j d^i &= d^i d^{j-1} \quad (i < j) \\ s^j d^i &= d^i s^{j-1} \quad (i < j) \\ &= \mathbb{1} \quad (i = j, j+1) \\ &= d^{i-1} s^j \quad (i > j+1) \\ s^j s^i &= s^{i-1} s^j \quad (i > j). \end{aligned}$$

The fact that $\delta^2 = 0$ follows formally from these identities. Now define $\rho = \sum_{i=0}^{n}(-1)^i s^i$. This is going to be our contracting homotopy:

$$\begin{aligned} \rho\delta + \delta\rho &= \sum_{j=0}^{n+1}\sum_{i=0}^{n+1}(-1)^{j+i}s^j d^i + \sum_{j=0}^{n}\sum_{i=0}^{n}(-1)^{i+j}d^j s^i = \sum_{j=1}^{n+1}\sum_{i=0}^{j-1}(-1)^{j+i}d^i s^{j-1} \\ &\quad + \cancel{\sum_{j=0}^{n}\sum_{i=j}^{j+1}(-1)^{j+i}} + (-\mathbb{1})^{2n+2} + \sum_{j=0}^{n-1}\sum_{i=j+2}^{n+1}(-1)^{j+i}d^{i-1}s^j + \sum_{j=0}^{n}\sum_{i=0}^{n}(-1)^{i+j}d^j s^i \\ &= \sum_{i=0}^{n}\sum_{j=i}^{n}(-1)^{j+1+i}d^i s^j + \sum_{i=1}^{n}\sum_{j=0}^{i-1}(-1)^{j+i+1}d^i s^j + \sum_{j=0}^{n}\sum_{i=0}^{n}(-1)^{i+j}d^j s^i + \mathbb{1} \\ &= \mathbb{1}. \end{aligned}$$

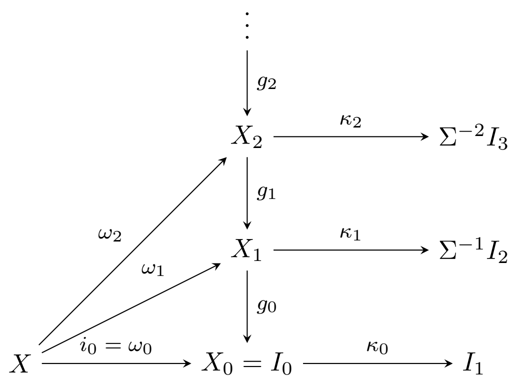
The bottom line is that the chain complexes (or sequences of spectra) constructed from the cosimplicial objects end up being exact. This completes the proof. ■

Now we are ready to construct the Adams spectral sequence.

2. The Construction

This should be rather painless... I hope. First we define the notion of *E-Adams tower*, which resembles the Postnikov tower. This tower will give rise to an exact couple, which will produce the Adams spectral sequence. Before constructing the exact couple, we will show that an E-Adams tower can be reconstructed from an E-resolution.

Definition 2.1. *A diagram of the following form*



will be called a tower, if the sequences $X_{n+1} \longrightarrow X_n \longrightarrow \Sigma^{-n} I_{n+1}$ *are fiber sequences. We can derive a sequence of the following form out of the tower:*

$$* \longrightarrow X \xrightarrow{i_0} I_0 \xrightarrow{i_1} I_1 \xrightarrow{i_2} \cdots$$

where i_n *is the composition* $I_n \longrightarrow \Sigma^n X_n \longrightarrow I_{n+1}$*. If the resulting sequence is an* E*-resolution, then we call the tower an* E*-Adams tower.*

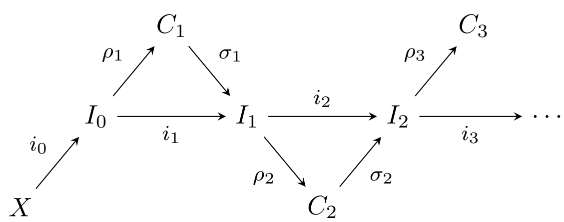
Remark 2.2. One thing that we can infer from any tower, is the fact $i_{s+1}i_s \simeq *$. It follows from the following sequence,

$$I_n \longrightarrow \Sigma^n X_n \longrightarrow I_{n+1} \longrightarrow \Sigma^{n+1} X_{n+1} \longrightarrow I_{n+2}$$

The part of the sequence that is labeled green is a fiber sequence, hence the composition is null.

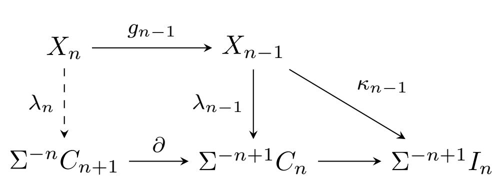
Proposition 2.3. *Every* E*-resolution gives rise to an* E*-Adams tower.*

Proof. Suppose that we have an E-resolution of X as in 2.1. We can break this resolution into syzygies,

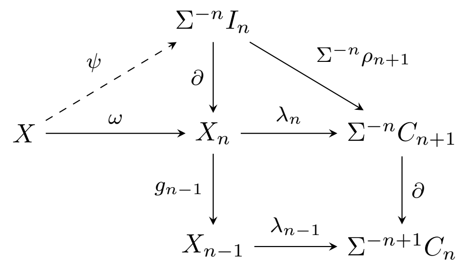


We define ρ_1 to be the cofiber of i_0, and let σ_1 be an induced map, such that $\sigma_1\rho_1 = i_1$. One can easily show that ρ_1 is E-epi and σ_1 is E-mono. Now if we show that $i_2\sigma_1 \simeq *$, we can iterate the construction and construct the syzygy. Since, $i_2\sigma_1\rho_1 \simeq *$, we conclude that $i_2\sigma_1 \simeq *$, for ρ_1 is E-epi and I_2 is E-injective.

Now we construct the E-Adams tower inductively. Suppose that we have constructed the tower up to the level $n-1$, and it satisfies the following properties: (a) $\kappa_{n-1} = (\Sigma^{-n+1}\sigma_n)\lambda_{n-1}$, where $\lambda_{n-1} : X_n \longrightarrow \Sigma^{-n+1}C_n$; (b) $\lambda_{n-1}\omega_{n-1} \simeq *$. Now let X_n be the fiber of κ_{n-1} (we don't really have a choice here). Define λ_n via the following diagram:



Thus, we can define κ_n to be $(\Sigma^{-n}\sigma_{n+1})\lambda_n$. We are left to construct ω_n, such that $\lambda_n\omega_n \simeq *$. Note, $\kappa_{n-1}\omega_{n-1} = (\Sigma^{-n+1}\sigma_n)\lambda_{n-1}\omega_{n-1} \simeq *$. This implies that there is a map $\omega : X \longrightarrow X_n$, such that $\omega_{n-1} = g_{n-1}\omega$. Then let us look at the diagram,



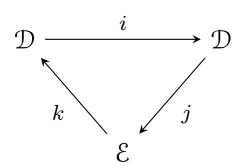
Notice that $\partial\lambda_n\omega = \lambda_{n-1}g_{n-1}\omega = \lambda_{n-1}\omega_{n-1} \simeq *$. Thus, there is ψ, such that $\lambda_n\omega = (\Sigma^{-n}\rho_{n+1})\psi$. Define ω_n to be $\omega - \partial\psi$. Let's check: $\lambda_n\omega_n = \lambda_n(\omega - \partial\psi) = \lambda_n\omega - \lambda_n\partial\psi = \lambda_n\omega - (\Sigma^{-n}\rho_{n+1})\psi = 0$. This finishes the inductive step of the construction of the tower. One can easily check that associated sequence to this tower is the one that we started off with. ■

Notice that we can extract another sequence from the E-Adams tower:

$$X = C_0 \xleftarrow{\gamma_0} \Sigma^{-1}C_1 \xleftarrow{\gamma_1} \Sigma^{-2}C_2 \xleftarrow{\gamma_2} \cdots$$

All the maps are the boundary maps of the appropriate cofiber sequences. We will refer to this sequence as *the associated inverse sequence.* Note that after smashing this sequence with E all the maps become trivial. Note also that the cofiber of γ_n is $\Sigma^{-n}I_n$. This sequence is what Ravenel calls an E-Adams resolution in [**RGB**].

Suppose we are given an E-Adams tower. To get the spectral sequence we construct an exact couple,



where $\mathcal{D} = \bigoplus_{s,t}\pi_{t-s}(X_s)$, $\mathcal{E} = \bigoplus_{s,t}\pi_t(I_s)$ are double graded groups. The maps are defined as follows,

$$i : \mathcal{D}^{s+1,t+1} = \pi_{t-s}(X_{s+1}) \xrightarrow{\pi_{t-s}(g_s)} \pi_{t-s}(X_s) = \mathcal{D}^{s,t}$$

$$j : \mathcal{D}^{s,t} = \pi_{t-s}(X_s) \xrightarrow{\pi_{t-s}(\kappa_s)} \pi_{t-s}(\Sigma^{-s}I_{s+1}) = \mathcal{E}^{s+1,t}$$

$$k : \mathcal{E}^{s+1,t} = \pi_{t-s}(\Sigma^{-s}I_{s+1}) \xrightarrow{\pi_{t-s}(\partial_s)} \pi_{t-s}(\Sigma X_{s+1}) = \mathcal{D}^{s+1,t}$$

It is not difficult at all to understand that we have an exact couple here. Thus, we obtain an exact sequence $(\mathcal{E}_r, d_r)$, which we call the *Adams spectral sequence.*

Remark 2.4. Before going into the next section let us comment on the grading of the differentials. With care and the use of induction one can show that in r-th derived couple the maps have the following grading: $i_r : \mathcal{D}_r^{s+1,t+1} \longrightarrow \mathcal{D}_r^{s,t}$, $j_r : \mathcal{D}_r^{s,t} \longrightarrow \mathcal{E}_r^{s+r,t+r-1}$ and $k_r : \mathcal{E}_r^{s,t} \longrightarrow \mathcal{D}_r^{s,t}$. Thus, $d_r : \mathcal{E}_r^{s,t} \longrightarrow \mathcal{E}_r^{s+r,t+r-1}$. Note that $\mathcal{E}^{s,t} = \pi_t(I_s) = 0$ if $s < 0$, which implies that $\mathcal{E}_r^{s,t} = 0$ if $s < 0$. If $r > s$, the differentials entering into $\mathcal{E}_r^{s,t}$ are all clearly 0. Thus, we see that $\mathcal{E}_{s+1}^{s,t} \supset \mathcal{E}_{s+2}^{s,t} \supset \ldots$, and $\mathcal{E}_\infty^{s,t} = \bigcap_{r>s}\mathcal{E}_r^{s,t}$. Notice also that $\mathcal{E}_r^{s,t}$ may not ever stabilize to $\mathcal{E}_\infty^{s,t}$.

There is an alternate exact couple that gives rise to the same spectral sequence. This exact couple actually arises from the associated inverse sequence. We'll write $K_n = \Sigma^{-n}C_n$. Replace $\mathcal{D}$ with $\mathcal{F} = \bigoplus_{s,t}\pi_{t-s}(K_s)$.

The maps are defined as follows:

$$i : \mathcal{F}^{s+1,t+1} = \pi_{t-s}(K_{s+1}) \xrightarrow{\pi_{t-s}(\gamma_s)} \pi_{t-s}(K_s) = \mathcal{F}^{s,t}$$

$$j : \mathcal{F}^{s,t} = \pi_{t-s}(K_s) \xrightarrow{\pi_{t-s}(\lambda_s)} \pi_{t-s}(\Sigma^{-s} I_s) = \mathcal{E}^{s,t}$$

$$k : \mathcal{E}^{s,t} = \pi_{t-s}(\Sigma^{-s} I_s) \xrightarrow{\pi_{t-s}(\Sigma^{-s}\rho_{s+1})} \pi_{t-s}(\Sigma K_{s+1}) = \mathcal{F}^{s+1,t}$$

where ρ_s was defined in the proof of 2.3. There are maps from $\nu_s : K_s \longrightarrow X_s$, such that $\kappa_s \nu_s = \Sigma^{-s}(\sigma_{s+1})$, $g_s \nu_{s+1} = \gamma_s \nu_s$ and $(\Sigma^{-s+1}\rho_s)\nu_s = \partial_s$. This produces a map between the exact couples that induces the identity map on the $\mathcal{E}_1$-page. We will use this exact couple to prove statements about the convergence of the Adams spectral sequence.

3. The $\mathcal{E}_2$-Page & The Convergence

In order to have a nice spectral sequence with nice $\mathcal{E}_2$-page and decent convergence, we need some assumptions on our ring spectrum. We will state the assumptions later. Under the assumptions on the ring spectrum, we can show that E-completion of spectra exist and the functor Ext makes sense. All we need to know is that $H\mathbb{Z}_{/p}$ and BP satisfy those conditions. We'll talk about the conditions later.

Definition 3.1. *If we have a sequence*

$$X_0 \xleftarrow{f_0} X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} \cdots$$

then the homotopy limit of the sequence, $\underleftarrow{\text{holim}}\, X_s$*, is the fiber of the following map* $\prod X_i \longrightarrow \prod X_j$*, where* $(j-1)$*-th component of the map is* $p_{j-1} - f_j p_j$*.*

Definition 3.2. *An E-Adams tower is simple if the associated inverse sequence has a trivial homotopy limit, i.e.* $\underleftarrow{\text{holim}}\, K_n = *$.

Definition 3.3. *An E-completion of a spectrum X is another spectrum $\widehat{X}$, with a map $X \longrightarrow \widehat{X}$ that induces an isomorphism on E_*-homology and $\widehat{X}$ has a simple E-Adams tower.*

Before stating the main theorem, let me mention something about E_*E that we will need. If we assume that E_*E is flat over $\pi_*(E)$, then the pair $(\pi_*(E), E_*E)$ is a graded Hopf algebroid. This assumption on E_*E will be referred to as *flatness of* E. To specify the Hopf algebroid structure, we need to provide the structure maps – $\eta_L, \eta_R : \pi_*(E) \longrightarrow E_*E$, $\Delta : E_*E \longrightarrow E_*E \otimes_{\pi_*E} E_*E$, $\varepsilon : E_*E \longrightarrow \pi_*(E)$, and $c : E_*E \longrightarrow E_*E$. We can define some of these maps right away: $\eta_L = \pi_*(e \wedge \mathbb{1})$, $\eta_R = \pi_*(\mathbb{1} \wedge e)$, $\varepsilon = \pi_*(\mu)$, and $c = \pi_*(\tau)$, where $\tau : E \wedge E \longrightarrow E \wedge E$ denotes the twist map. To define Δ we'll need the following lemma, which will also be used later.

Lemma 3.4. *There is a natural map*

$$E_*E^{\otimes n} \otimes_{\pi_*(E)} E_*(X) \longrightarrow \pi_*(E^{\wedge(n+1)} \wedge X),$$

which an isomorphism.

Proof. We will define the map in due course. Actually, let's look at the case $n = 1$. We define

$$m : E_*E \otimes_{\pi_*(E)} E_*(X) \longrightarrow \pi_*(E^{\wedge 2} \wedge X),$$

so that if $\alpha \in \pi_*(E \wedge E)$, $\beta \in \pi_*(E \wedge X)$, then $m(\alpha \otimes \beta) = (\mathbb{1} \wedge \mu \wedge \mathbb{1})(\alpha \wedge \beta)$. This map is an isomorphism for the following reason. The functors $E_*E \otimes_{\pi_*(E)} E_*(-)$ and $\pi_*(E^{\wedge 2} \wedge -) = (E \wedge E)_*(-)$ are homology theories that agree on S^0 via m (m can be thought of as a map between homology theories). Thus, they ought to be naturally isomorphic via m. Note that we implicitly used the flatness of E to conclude that $E_*E \otimes_{\pi_*(E)} E_*(-)$ is a homology theory. The construction of the rest of the isomorphisms is done via induction. ■

We define Δ as $\pi_*(\mathbb{1} \wedge e \wedge \mathbb{1}) : \pi_*(E \wedge E) = E_*E \longrightarrow \pi_*(E \wedge E \wedge E) \cong E_*E \otimes_{\pi_*(E)} E_*E$. It is a routine check to verify the axioms of Hopf algebroid.

Theorem 3.5. *If E is flat, and X has an E-completion, the spectral sequence $(\mathcal{E}_r, d_r)$ for X converges to $\pi_*(\widehat{X})$ and*

$$\mathcal{E}_2^{s,t} = \operatorname{Ext}^{s,t}_{E_*E}(\pi_* E, E_*(X)),$$

where $\operatorname{Ext}^{s,t}_{\Gamma}(M, -)$ denotes the t-th graded piece of the s-th derived functor of $\operatorname{Hom}_{\Gamma}(M, -)$ over the category of graded left Γ-comodules.

It actually does make sense to talk about derived functors, since, due to flatness of E, the category of graded left E_*E-comodules is abelian. We will discuss the filtration of $\pi_*(\widehat{X})$ in due course.

Remark 3.6. We can show that $\mathcal{E}_2$-page has the above-mentioned form after a discussion on "uniqueness" of Adams spectral sequence. Recall that we have constructed the Adams spectral sequence from the E-Adams tower, and there could be lots of them, and in principle, they may give us different spectral sequences. Theorem 3.5 hints us that starting from $\mathcal{E}_2$-page, the spectral sequences must be isomorphic. This is what we mean by "uniqueness" of the spectral sequence. Let $\mathcal{T}^1_X$ and $\mathcal{T}^2_X$ be two E-Adams towers for X. These two towers have corresponding E-resolutions, $\mathcal{R}^1_X$ and $\mathcal{R}^2_X$ for X. Lifting the identity, provides us with a chain homotopy equivalence $\rho : \mathcal{R}^1_X \longrightarrow \mathcal{R}^2_X$. We can lift this map to a map of towers $\tau : \mathcal{T}^1_X \longrightarrow \mathcal{T}^2_X$. This induces a map between the corresponding spectral sequences, $\tau_* : \mathcal{E}^{(1)}_* \longrightarrow \mathcal{E}^{(2)}_*$. Now lets see what τ_* does on $\mathcal{E}_1$-page. The differentials are easy to compute:

$$\begin{array}{ccccc} \mathcal{E}_1^{s,t} & \longrightarrow & \mathcal{D}_1^{s,t} & \longrightarrow & \mathcal{E}_1^{s+1,t} \\ \| & & \| & & \| \\ \pi_t(I_s) & \longrightarrow & \pi_{t-s}(X_{s+1}) & \longrightarrow & \pi_t(I_{s+1}) \end{array}$$

$$d_1 = \pi_t(j_{s+1}) : \pi_t(I_s) \longrightarrow \pi_t(I_{s+1})$$

The induced maps between the $\mathcal{E}_1$-pages are, $\pi_t(\rho_s) : \pi_t(I^{(1)}_s) \longrightarrow \pi_t(I^{(2)}_s)$. Since ρ is chain homotopy equivalence, then so is $\pi_t(\rho_*)$ for all t. Thus, $\pi_t(\rho_*)$ induces an isomorphism on cohomology of $\pi_t(I^{(k)}_*)$. However, $\mathcal{E}_2^{(k)s,t} = H^s(\pi_t(I^{(k)}_*))$. A similar discussion proves that any map $X \longrightarrow Y$ induces a natural map between the spectral sequences, "modulo $\mathcal{E}_1$-page". This statement applies in general and no assumptions were needed on E other than the ones made in section 2.

From the previous remark we can conclude that we can use any E-resolution to compute $\mathcal{E}_2$-page of the Adams spectral sequence. We will use the one that we already know of, i.e. the canonical resolution. The $\mathcal{E}_1$-page look as follows, $\mathcal{E}_1^{s,*} = \pi_*(I_s) = \pi_*(E^{\wedge(s+1)} \wedge X) = E_*E^{\otimes s} \otimes_{\pi_*(E)} E_*(X)$. After some examination one realizes that the sequences we get on $\mathcal{E}_1$-page

$$0 \longrightarrow E_*(X) \longrightarrow E_*E \otimes_{\pi_*(E)} E_*(X) \longrightarrow E_*E^{\otimes 2} \otimes_{\pi_*(E)} E_*(X) \longrightarrow \cdots$$

is the cobar complex. The cohomology of this complex is known to be $H^*(C(E_*(X))) = \operatorname{Ext}^{*,*}_{E_*E}(\pi_*(E), E_*(X))$. It is worked out in [**RGB**, A1.2.11, A1.2.12].

Now we'll discuss the convergence of the Adams spectral sequence. Recall that $X \longrightarrow \widehat{X}$ induces an isomorphism on E_*-homology. This implies that $E \wedge X \longrightarrow E \wedge \widehat{X}$ is an equivalence. The map $X \longrightarrow \widehat{X}$ induces a (natural) map between the canonical resolutions. Thus, we obtain a map from one spectral sequence to the other one $\mathcal{E}_* \longrightarrow \widehat{\mathcal{E}}_*$. On page 1, the map looks as follows, $\mathcal{E}_1^{s,t} = \pi_t(E^{\wedge(s+1)} \wedge X) \longrightarrow \pi_t(\widehat{E}^{\wedge(s+1)} \wedge \widehat{X}) = \widehat{\mathcal{E}}_1^{s,t}$. This is clearly an isomorphism. Therefore, $\mathcal{E}_* \longrightarrow \widehat{\mathcal{E}}_*$ is an isomorphism. If we pick any two resolutions of X and $\widehat{X}$, respectively, we are guaranteed by 3.6, that their spectral sequences are isomorphic, in a natural way, starting from page 2. Thus, we will study the convergence of the spectral sequence of $\widehat{X}$.

It will be convenient to phrase the convergence in terms of proposition. There, we will also specify the filtration of $\pi_*(\widehat{X})$.

Proposition 3.7. *Suppose $\widehat{X}$ has a simple E-Adams tower. Then*

$$\mathcal{E}_\infty^{s,t} \cong \operatorname{im} \pi_{t-s}(K_s) / \operatorname{im} \pi_{t-s}(K_{s+1})$$

where the images are taken in $\pi_{t-s}(\widehat{X})$*, and* $\bigcap \operatorname{im} \pi_*(K_n) = 0$.

Proof. We first show that the intersection of the filtration pieces is 0. Let $\widehat{X} \longleftarrow K_1 \longleftarrow K_2 \longleftarrow \dots$ be the inverse sequence associated to a simple E-Adams tower of $\widehat{X}$. Then, by definition, $\underleftarrow{\operatorname{holim}}\, K_n = *$. This implies, among other things, that $\underleftarrow{\lim}\, \pi_*(K_n) = 0$. We will write for the L_n for $\bigcap \operatorname{im} \pi_*(K_{n+r}) \subset \pi_*(K_n)$. Then we have a sequence, $L_0 \longleftarrow L_1 \longleftarrow L_2 \longleftarrow \dots$, where the maps are the restrictions of $\pi_*(\gamma_s)$'s. These restrictions are surjective. We are trying to show that $L_0 = 0$. Suppose that $x_0 \in L_0$. Then there is $x_1 \in L_1$, such that it maps to x_0. Similarly, there is an element $x_2 \in L_2$, that maps to x_1. If we continue this way, we obtain an element $(x_0, x_1, x_2 \dots)$ of $\underleftarrow{\lim}\, \pi_*(K_n)$. However, this element must be 0, since the inverse limit is trivial. This implies that $x_0 = 0$.

I'll define a map $\eta : \mathcal{E}^{s,t}_\infty \longrightarrow \mathcal{G}^{s,t}/\mathcal{G}^{s+1,t+1}$, where $\mathcal{G}^{s,t} = \operatorname{im} \pi_{t-s}(K_s) \subset \pi_{t-s}(\widehat{X})$. Let $[\alpha] \in \mathcal{E}^{s,t}_\infty$, where $\alpha \in \mathcal{E}^{s,t}$ is an element that represents $[\alpha]$. We would like to show that $k(\alpha) = 0$. Notice that if $r > s$, then $d_r([\alpha]) = 0$. Recall that $d_r = j_r k_r$; thus, $k_r([\alpha]) = k(\alpha) \in \ker j_r = \operatorname{im} i_r$. If we take into account the grading we can show that $k(\alpha) \in \operatorname{im} \pi_{t-s-1}(K_{s+r}) \subset \pi_{t-s-1}(K_{s+1})$. Thus, $k(\alpha) \in \bigcap_{r>s} \operatorname{im} \pi_{t-s-1}(K_{s+r}) = 0$, which shows the claim.

Thus, by exactness of the sequence $\mathcal{F}^{s,t} \longrightarrow \mathcal{E}^{s,t} \longrightarrow \mathcal{F}^{s+1,t}$, we see that there is $\beta \in \mathcal{F}^{s,t}$, such that $j(\beta) = \alpha$. There is a quotient map $\varphi : \mathcal{F}^{s,t} \longrightarrow \mathcal{G}^{s,t} \longrightarrow \mathcal{G}^{s,t}/\mathcal{G}^{s+1,t+1}$. Thus, we define $\eta([\alpha]) = \varphi(\beta)$. We need to show that the definition is independent of the choice of α and β. This is equivalent to stating that if $[\alpha] = 0$, then for any lift β of α, $\varphi(\beta) = 0$. I would like to show first that in this case the choice of β does not matter. Choose $\overline{\beta}$, such that $j(\overline{\beta}) = \alpha$. Then $j(\beta - \overline{\beta}) = 0$, which implies that $\beta - \overline{\beta} \in \operatorname{im} i$, implying that $\varphi(\beta) = \varphi(\overline{\beta})$. The fact that $[\alpha] = 0$ implies that $[\alpha] \in \operatorname{im} d_r$ for some $r \le s$. One can show by induction that the set of α's in $\mathcal{E}^{s,t}$ that satisfy this property is $j(h^{-1}(k(\mathcal{E}^{s-r,t-r+1})))$, where the maps are shown in the diagram:

$$\begin{array}{ccccc} & & \pi_{t-s}(K_s) & \xrightarrow{\quad j \quad} & \mathcal{E}^{s,t} \\ & & \Big\downarrow h & & \\ \mathcal{E}^{s-r,t-r+1} & \xrightarrow{\ k\ } & \pi_{t-s}(K_{s-r+1}) & & \end{array}$$

Thus, we can find $\gamma \in \mathcal{E}^{s-r,t-r+1}$ and $\beta \in \pi_{t-s}(K_s)$, such that $h(\beta) = k(\gamma)$ and $j(\beta) = \alpha$. Then $ih(\beta) = ik(\gamma) = 0$, implying that $\varphi(\beta) = 0$, since $\pi_{t-s}(K_s) \longrightarrow \pi_{t-s}(X)$ factors through ih.

Now suppose that $[\alpha] \neq 0$. If $\eta([\alpha]) = 0$, then there is a maximal r, such that the image of β, λ, in $\pi_{t-s}(K_{s-r+1})$ that is nonzero. Then $i(\lambda) = 0$. That means there is $\gamma \in \mathcal{E}^{s-r,t-r+1}$, such that $k(\gamma) = \lambda$. This implies that $d_r([\gamma]) = [\alpha]$, thus, contradicting the non-triviality of $[\alpha]$. This shows that η is injective.

Now let's show that η is surjective. Suppose we are given $\alpha = j(\beta) \in \mathcal{E}^{s,t}$. This equivalent to saying that $k(\alpha) = 0$. If α survives (i.e. is a cycle) up till $(r-1)$-th page, then $k_r([\alpha]) = k(\alpha) = 0$. This implies that $d_r([\alpha]) = 0$. Thus, α is cycle on r-th page as well. This proves that α defines a class in $\mathcal{E}^{s,t}_\infty$. Clearly, $\eta([\alpha]) = \varphi(\beta)$, which proves the surjectiveness. ■

The convergence may look a bit weird. However, it ends up being nice in the cases we may be interested in. Actually, one thing we need is the existence of the E-completion. These assumptions are taken from [**RGB**], and they guarantee the existence of the E-completion.

Assumptions 3.8. *(a) E is commutative and associative.*

(b) E is connective, i.e. $\pi_r(E) = 0$ for $r < 0$.

(c) The map $\mu_* : \pi_0(E) \otimes \pi_0(E) \longrightarrow \pi_0(E)$ is an isomorphism.

(d) Let $\theta : \mathbb{Z} \longrightarrow \pi_0(E)$ be the unique ring homomorphism, and let $R \subset \mathbb{Q}$ be the largest subring to which θ extends. Then $H_r(E; R)$ is finitely generated over R for all r.

Theorem 3.9. *If X is connective and E satisfies the conditions in 3.8, then $\widehat{X} = XG$, if $\pi_0(E) = G$ for the cases $G = \mathbb{Q}$, $\mathbb{Z}_{(p)}$, and $\mathbb{Z}$. If $G = \mathbb{Z}_{/p}$ and $\pi_*(X)$ are finitely generated, then $\widehat{X} = X\mathbb{Z}_p$, where $\mathbb{Z}_p$ denotes the p-adic integers.*

Proof. [**ABB**, 14.6, 15.∗]. ■

Let me comment on the Adams spectral sequence for $E = H\mathbb{Z}_{/p}$. The $\mathcal{E}_2$-page, is that of classical Adams spectral sequence, $\operatorname{Ext}^{*,*}_{\mathcal{A}_{p*}}(\mathbb{Z}_{/p}, H_*(X; \mathbb{Z}_{/p}))$. Where $\mathcal{A}_{p*}$ is the dual Steenrod algebra. If X is finite CW-spectrum, the spectral sequence converges to $\pi_*(X) \otimes \mathbb{Z}_p$, which is $\pi_*(X)$ modulo the non-p-torsion.

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