Outline:

- Degree 1 & linear functors
  (function story)
- Examples of
- Derivatives & Layers of Taylor tower
- Obtaining $P_1F(X)$
- Some calculations using $F = \mathrm{Id}$

Rmk: $\left\{ \begin{array}{l} \text{m'fld calc studies } \underline{\text{contra}}\text{variant functors} \\ \text{htpy } \text{'' '' } \underline{\text{co}}\text{variant '' ''} \end{array} \right.$

Recall: fxn $f: \mathbb{R} \circlearrowleft$ is <u>linear</u> if $f(x+y) = f(x) + f(y)$
<u>deg 1</u> if $f(x+y) = f(x) + f(y) + f(0)$

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & x \arrow[d] \\ y \arrow[r] & x+y \end{tikzcd} \xrightarrow{\ f\ } \underbrace{\begin{tikzcd} f(0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & f(x) \arrow[d] \\ f(y) \arrow[r] & f(x+y) \end{tikzcd}}_{\text{deg } 1} = \begin{tikzcd} f(0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & f(x)f(y) \arrow[d] \\ \bullet \arrow[r] & f(x+y) \end{tikzcd}$$

$\Rightarrow f(x)f(y) - f(x+y) = f(0)$

if Ab,

$\Rightarrow f(x) + f(y) - f(x+y) = f(0)$

if red'd

$\Rightarrow f(x) + f(y) - f(x+y) = 0$

A word on spectra

Ω^∞-spaces $\leftrightsquigarrow$ spectra
"abelian spaces"

in Ab, $\times = +$] – this lets us tell our story

Rmk made @ conference:

$$f(x) \cdot f(y) - f(x+y) = f(0)$$

have two choices of interp for each of these operations.

Choices made yield different Calculi

cf: Randy's Dual Calculus.

Examples

- (Reduced) homology theories are linear "co rep" by some spectrum C

i.e. $h_*(X) = X \wedge C$ $\quad (* \wedge C = *$, so red'd$)$

looks like $f(x) = cx$

FACT: up to some fiddling, all linear functors are of this form

RMK How do I get the analog of $f(x) = cx + b$?

$\boxed{\text{Pg } \cancel{8.19}}$ 3

First need extra language

$\boxed{\text{Derivs \& Layers}}$

$$\left.\begin{aligned} P_0 f(x) &= f(0) \\ P_1 f(x) &= f(0) + f'(0)\cdot x \\ P_1 f(x) - P_0 f(x) &= f'(0)\cdot x \end{aligned}\right] \mathbb{R}$$

$(\Omega^\infty(C_1 \wedge X))$

$$\underbrace{\text{hofib}(P_1 F(x) \twoheadrightarrow P_0 F(x))}:= \underbrace{D_1 F(x)}_{1^{st}\text{ layer}} \text{ “=” } C_1 \wedge X$$

0^{th} layer $= P_0 F(x) = F(0)$

~~Ex I P1~~

$\boxed{\text{Def}}$ We say a tower splits if $=$ product of layers

If $\underbrace{P_1 F(x)\text{'s tower}}$ splits,

$$\begin{array}{ccc} C_1 \wedge X & \to & P_1 F(x) \\ & & \downarrow \\ F(0) & \to & P_0 F(x) = F(0) \end{array}$$

then $P_1 F(x) \sim (C_1 \wedge X)\cdot F(0)$

$= (C_1 \wedge X) + F(0)$

$\uparrow$ if in Spectra

Analog to

$f(x) = cx + b$

Back to layers

In general,

$$P_n f(x) - P_{n-1} f(x) = \frac{f^{(n)}(0)}{n!} x^n \quad \Big] \ \mathbb{R}$$

$\mathbb{R}$		GC, Spectra
x^n	$\rightsquigarrow$	$\underbrace{X \wedge \cdots \wedge X}_{n} = X^{\wedge n}$
$f^{(n)}(0)$	$\rightsquigarrow$	spectrum C_n,
div by $n!$	$\rightsquigarrow$	homotopy orbits of Σ_n action on $X^{\wedge n}$

So $D_n F(X) = (C_n \wedge X^{\wedge n})_{h\Sigma_n}$

or

$= \Omega^\infty (C_n \wedge X^{\wedge n})_{h\Sigma_n}$ (to land in spaces)

Also,

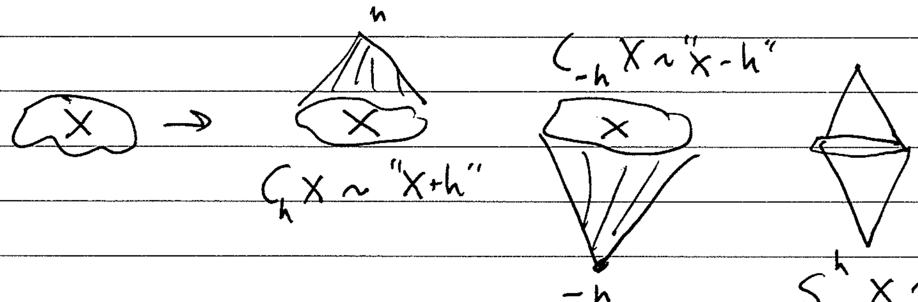
$D_n F(X) = \mathrm{hofib}(P_n F(X) \twoheadrightarrow P_{n-1} F(X))$

"nth layer"

P5

$P_1F(X)$ / Linearization

Recall: $\overset{\mathbb{R}}{\Delta f(x)} \approx T_1 f(x) = \dfrac{-f(x-h) + f(x) + f(x+h)}{\cancel{2h}}$ forget about this



so $T_1F(X) \text{ "}= -F(S^h_{-h}X) + F(C_{-h}X) + F(C^h_h X)\text{"}$

i.e.

$$\begin{tikzcd} T_1F(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & F(CX) \arrow[d] \\ F(CX) \arrow[r] & F(SX) \end{tikzcd}$$

If $F(\cdot) \sim \cdot$, then

$$\begin{tikzcd} T_1F(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \cdot \arrow[d] \\ \cdot \arrow[r] & F(SX) \end{tikzcd} \Rightarrow \Omega F(\Sigma X) \sim T_1F(X)$$

To obtain better linear approximation, iterate

$T_1^2 F(x) := T_1(T_1 f(x))$

$T_1^3 F(x) := T_1(T_1(T_1 F(x)))$

Assemble / "limit" over process:

$$P_1 F(X) \simeq \operatorname{hocolim}_i T_1^i F(X) \underset{\text{if } F \text{ red'd}}{\simeq} \Omega^\infty F(S^\infty X)$$

$\boxed{Ex}$ Id is red'd

$\therefore\ P_1 Id(X) \simeq \Omega^\infty S^\infty X = Q(X)$

If functor red'd, $D_1 F(X) = P_1 F(X)$

$\therefore\ D_1 Id(X) = \Omega^\infty(\Sigma^\infty X) = \Omega^\infty(C_1 \wedge X)$

(up to some conn. issues ... need X 2-conn)

Coeff of $D_1 = \partial_1 Id(\circ) = C_1$ s.t. $C_1 \wedge X = \Sigma^\infty X$

i.e. $= \mathbb{S}$ (or $\mathbb{S}^0$)

THAT IS, 1^{st} deriv of Id = sphere spectrum

Rmk: $D_2 Id(X) \simeq \Omega Q((X \wedge X)_{h\Sigma_2})$

TAYLOR TOWER of Id is gnarly

- Brenda Johnson – (thesis) (under Goodwillie)
- Arone & Kankaanrinta – "Func Model for It. Snaith splitting"
- Ching – Bar const for top operads (thesis)
- Arone & Mahowald – "Goodwillie tower of Id functor"