Math 225 - Practice Midterm 1 - SOLUTIONS

1. **True or False**?

(a) The set of vectors $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$ is linearly independent.

FALSE. Any set of vectors which includes the zero vector is always **dependent**.

(b) The set of vectors $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ is linearly independent.

FALSE. This is equivalent to asking whether there are any nontrivial solutions to the corresponding homogeneous system of equations. Row reduction gives

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 2 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The vector $(-1\;1\;1)^T$ is a solution, which means that

$$-\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{0}$$

(c) The set of vectors $\left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ is linearly independent.

TRUE. Again, we row reduce the resulting matrix to find

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 2 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

There are no free variables, so the system has no nontrivial solution. We conclude that the given set of vectors is linearly independent.

(d) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible, then a cannot be 0.

FALSE. The permutation matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is invertible (it is its own inverse).

(e) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible, then b cannot be 0.

FALSE. The identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is of course invertible ($I^{-1} = I$).

(f) A homogeneous system of equations may have no solution.

FALSE. A homogeneous system of equations always has *at least* the trivial solution. It may or may not have others.

(g) If the columns of A are independent, then A must have at least as many rows as columns.

TRUE. This was one of the results mentioned in class. We know that the columns are independent if there is a pivot in each column. But there cannot be more than one pivot in each row, so there must be at least as many rows as pivots, which is the same as the number of columns.

2. Consider the system of equations

$$\begin{aligned} w + x + y + z &= 1 \\ -w + y + z &= 1 \\ x + 2y + 3z &= 1 \end{aligned}$$

(a) Write down the corresponding augmented matrix and convert it into reduced row echelon form.

SOLUTION. The augmented matrix is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 1 \end{pmatrix}.$$

Row reducing to echelon form, we get

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 3 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

To further reduce to the rref, we first clear the entries above the third pivot

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

and above the second pivot

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 2 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 & -2 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

(b) Find the general solution to this system of equations.

SOLUTION. We read off from the rref matrix the solution

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y-2 \\ -2y+4 \\ y \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 0 \\ -1 \end{pmatrix} + y \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}$$

(c) Give a particular solution.

SOLUTION. The particular solution when $y = 0$ is $\begin{pmatrix} -2 \\ 4 \\ 0 \\ -1 \end{pmatrix}$.

3. Let

$$A = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 3 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 5 & 2 & 1 \\ -1 & 4 & 2 \\ -2 & -2 & 3 \end{pmatrix}.$$

Of the matrix products AB, BA, $A^T B^T$, $B^T A^T$, and BA^T, compute the ones that are well-defined.

SOLUTION. We have

$$AB = \begin{pmatrix} 13 & 2 & -3 \\ 2 & 14 & 7 \end{pmatrix},$$

BA and $A^T B^T = (BA)^T$ are not defined,

$$B^T A^T = (AB)^T = \begin{pmatrix} 13 & 2 \\ 2 & 14 \\ -3 & 7 \end{pmatrix},$$

and

$$BA^T = \begin{pmatrix} 7 & 11 \\ -8 & 11 \\ -5 & -8 \end{pmatrix}.$$

4. Find the inverse of the following matrices:

(a) $A = \begin{pmatrix} 2 & 5 \\ -1 & 1 \end{pmatrix}$

SOLUTION.

$$A^{-1} = \frac{1}{2 \cdot 1 - 5(-1)} \begin{pmatrix} 1 & -5 \\ 1 & 2 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 1 & -5 \\ 1 & 2 \end{pmatrix}.$$

(b) $B = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 2 & -1 & 1 \end{pmatrix}$

SOLUTION. We set up a big augmented matrix and reduce:

$$\begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{pmatrix} \overset{R2-2R1}{\sim} \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \end{pmatrix}$$

$$\overset{R2\leftrightarrow R3}{\sim} \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\overset{R3-2R2}{\sim} \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \\ 0 & 0 & -1 & 4 & 1 & -2 \end{pmatrix}$$

$$\overset{-R3}{\sim} \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & -4 & -1 & 2 \end{pmatrix}$$

$$\overset{R2-R3}{\sim} \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -4 & -1 & 2 \end{pmatrix}$$

$$\overset{R1+R2}{\sim} \begin{pmatrix} 1 & 0 & 0 & 3 & 1 & -1 \\ 0 & 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -4 & -1 & 2 \end{pmatrix}.$$

It follows that

$$B^{-1} = \begin{pmatrix} 3 & 1 & -1 \\ 2 & 1 & -1 \\ -4 & -1 & 2 \end{pmatrix}.$$

5. Show that if A is invertible, then so is A^T, with $(A^T)^{-1} = (A^{-1})^T$.

SOLUTION. Start with the equation $AA^{-1} = I$ and take the transpose of both sides. This gives

$$(A^{-1})^T A^T = I^T = I.$$

Since the matrix $(A^{-1})^T$ multiplies A^T to give the identity, it follows that

$$(A^{-1})^T = (A^T)^{-1}.$$