Tuesday, November 1 $\quad **\quad$ *Invertible matrices & the LU decomposition*

1. (Permutation matrices) Any matrix that can be obtained from the identity matrix by switching rows is called a **permutation matrix**. A **simple** permutation matrix is one obtained from the identity matrix by performing a single row switch.

   (a) Explain why any permutation matrix is a product of simple permutations.

   (b) Show that for any simple permutation matrix E, the inverse E^{-1} is simply E.

   (c) The equation $E^{-1} = E$ does not hold for all permutation matrices. For example, write the matrix
   $$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$
   as a product of simple permutations and use this to find P^{-1}. Show that P^{-1} is in fact P^2.

2. (Permuted LU decomposition) Not every matrix can be row-reduced into echelon form without switching any rows. A matrix A may need to be first multiplied by a permutation P in order to get an LU decomposition. For example, consider
   $$A = \begin{pmatrix} 0 & 3 & -1 \\ 2 & -2 & -1 \\ 1 & 2 & 0 \end{pmatrix}.$$

   (a) Let $P_{1,2}$ be the simple permutation corresponding to switching the first two rows. Find the LU decomposition for $P_{1,2}A$. Then
   $$A = P_{1,2}^{-1}LU$$
   is a permuted LU decomposition of A.

   (b) Use the permuted LU decomposition from (a) to solve $A\mathbf{x} = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$. Hint: just solve
   $$LU = P_{1,2}\begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}.$$

   (c) Let $P_{1,3}$ be the simple permutation corresponding to switching the first two rows. Find the LU decomposition for $P_{1,3}A$. Then
   $$A = P_{1,3}^{-1}LU$$
   is a permuted LU decomposition of A.

   (d) Use the permuted LU decomposition from (c) to solve $A\mathbf{x} = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}$.