**Tuesday, November 8** $\quad **\quad$ *Determinants*

1. Recall that if $A$ is an $n \times k$ matrix, then the transpose $A^T$ is the $k \times n$ matrix whose columns are the rows of $A$.

   (a) Show that if $A$ is $2 \times 2$, then $\det(A) = \det(A^T)$.

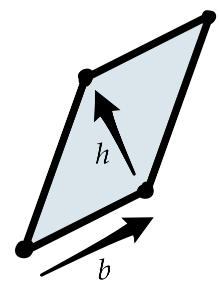
   (b) For a general $n \times n$ matrix $A$, consider the cofactor calculation of $\det(A^T)$ along row $i$. How would you describe this in terms of the original matrix? How is $\det(A^T)$ related to $\det(A)$?

   (c) Find the determinant of

$$A = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 2 & -5 & 0 & 0 \\ 7 & 14 & 2 & 0 \\ -5 & 8 & 9 & 1 \end{pmatrix}$$

2. An important geometric interpretation of determinants is in terms of area and volume. Recall that if $P$ is a parallelogram, then the area of $P$ can be calculated as

$$\text{Area} = (\text{base}) \cdot (\text{height}).$$




   (a) Let $A$ be a $2 \times 2$ matrix with columns given by $\mathbf{u}$ and $\mathbf{v}$. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are linearly *dependent* (in other words, one is a multiple of the other). What is $\det(A)$? Form the "parallelogram" with sides $\mathbf{u}$ and $\mathbf{v}$. What is the area of this parallelogram?

   (b) Suppose that $A$ is a $2 \times 2$ diagonal matrix with columns $\mathbf{u}$ and $\mathbf{v}$. What is the area of the resulting parallelogram, and what is $\det(A)$?

   (c) Suppose that $A$ is a lower triangular $2 \times 2$ matrix with columns $\mathbf{u}$ and $\mathbf{v}$. What are the base and height of the parallelogram? What is the area, and what is $\det(A)$?

   (d) Let $A$ be any $2 \times 2$ matrix with nonzero entry in position $(1,1)$. Then you can add an appropriate multiple of $\mathbf{u}$ to $\mathbf{v}$ in order to make $A$ into a lower triangular matrix. The determinant of $A$ is unchanged. Why is the area of the parallelograms also unchanged?

   (e) Finally, explain how to remove the assumption that the $a_{11} \neq 0$. That is, establish a relationship between $\det(A)$ and the area of the parallelogram determined by the columns of $A$, for any $A$.

3. Use the previous problem to find the area of the following parallelograms.

   (a) $P$ has vertices $(0,0)$, $(3,1)$, $(-1,2)$, $(2,3)$.

   (b) $P$ has vertices $(-1,0)$, $(0,5)$, $(1,-4)$, $(2,1)$. Hint: It may help to translate the parallelogram so that one of the vertices is the origin.