Lecture 37

Nov. 30 2011

Conflict Final: contact me ASAP

Quiz: Thursday (on Flux, Divergence Theorem)

Last time: Stokes' thm

$$\int_{\partial S} \vec{F}\cdot d\vec{r} = \iint_{S} \underbrace{(\nabla\times\vec{F})}_{\text{curl }\vec{F}}\cdot dS$$

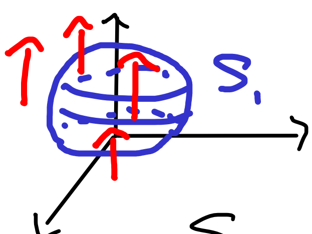
$\int_{\partial S} \vec{F}\cdot d\vec{r}$: work done by $\vec{F}$ on particle moving along ∂S

$\iint_{S} (\nabla\times\vec{F})\cdot dS$: flux of curl field across S.

We looked at $\vec{F} = (-y, x, 1)$ S_1 given by $x^2+y^2+(z-1)^2=2$, $z\geq 0$

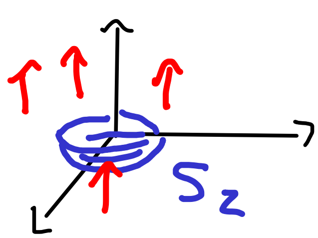
and S_2 given by $x^2+y^2+(z-1)^2=2$, $z\leq 0$.

Found $\nabla\times\vec{F} = (0,0,2)$, $\int_{\partial S_1} \vec{F}\cdot d\vec{r} = \iint_{S_1} (\nabla\times\vec{F})\cdot dS = 2\pi$.



Expect flux >0.

Same computation gave $\int_{\partial S_2} F\cdot d\vec{r} = \iint_{S_2} (\nabla\times\vec{F})\cdot dS = -2\pi$.

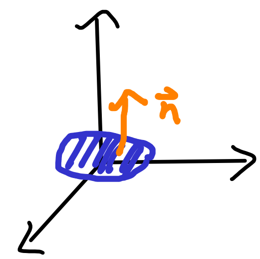


Fluid flowing into S_2 (against outward normal direction)

Expect flux <0.

Note that since $\int_{\partial S} \vec{F}\cdot d\vec{r}$ only depends on ∂S, then $\iint_{S_1} (\nabla\times\vec{F})\cdot dS = \iint_{S_3} (\nabla\times\vec{F})\cdot dS$

for any surface S_3 w/ same boundary (and orientation of boundary).



__Example__ S_3 = unit disc $x^2+y^2\le 1$, $z=0$

Then $\iint_{S_1} (\nabla\times\vec{F})\cdot dS = \iint_{S_3} (\nabla\times\vec{F})\cdot dS$

$= \iint_{disc} (0,0,2)\cdot \vec{k}\, dA = 2\ \text{area}(\text{disc}) = 2\pi.$

Another way to think of this: Div theorem says

$$\iint_S \vec{F}\cdot dS = \iiint_E \operatorname{div}\vec{F}\, dV.$$

So $\iint_S (\nabla\times\vec{F})\cdot dS = \iiint_E \operatorname{div}(\nabla\times\vec{F})\, dV$

$\nabla\times\vec{F} = (0,0,2)$ so $\operatorname{div}(\nabla\times\vec{F}) = 0.$

So RHS $=0$, but LHS $= 2\pi$.

What's the problem?

__Answer:__ S is __not__ the boundary surface of any solid.

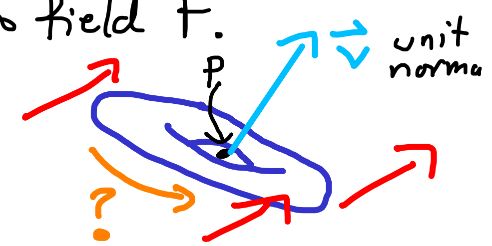
If $S = \partial E$, then ∂S is __empty__.

__Example__ E = solid sphere, $S=\partial E$ = sphere,

S does not have a boundary curve.

How can we see that curl $\vec{F}$ measures rotational movement?

Imagine inner tube, radius ε, located at P, moving according to field $\vec{F}$.



How much is it spinning?

Write C_ε = circle of radius ε, center at P

Tangential velocity given by $\vec{F} \cdot \vec{T}$ ← unit tangent.

$$\text{Average tangential velocity} = \frac{1}{2\pi\varepsilon} \int_{C_\varepsilon} \vec{F} \cdot \vec{T}\, ds = \frac{1}{2\pi\varepsilon} \int_{C_\varepsilon} \vec{F} \cdot d\vec{r}$$

But
$$\frac{\text{Average tangential velocity}}{2\pi\varepsilon} = \frac{\text{Angular velocity}}{2\pi}$$

so
$$\text{Angular velocity} = \frac{1}{\varepsilon}\,\text{Average tangential velocity}$$

$$= \frac{1}{\varepsilon} \frac{1}{2\pi\varepsilon} \int_{C_\varepsilon} \vec{F} \cdot d\vec{r} \underset{\text{Stokes}}{=} \frac{1}{2\pi\varepsilon^2} \iint_{D_\varepsilon} (\nabla \times \vec{F}) \cdot d\vec{S}$$

As $\varepsilon \to 0$, can approximate $\nabla \times \vec{F}$ on D_ε by $(\nabla \times \vec{F})(P)$.

So
$$\text{Angular velocity} \approx \frac{1}{2\pi\varepsilon^2} \iint_{D_\varepsilon} (\nabla \times \vec{F})(P) \cdot \vec{n}\, dS$$

$$= \frac{1}{2\pi\varepsilon^2} (\nabla \times \vec{F})(P) \cdot \vec{v} \overbrace{\iint_{D_\varepsilon} dS}^{\pi\varepsilon^2}$$

$$= \frac{1}{2} (\nabla \times \vec{F})(P) \cdot \vec{v}$$

This is largest when $\vec{v}$ points in direction $(\nabla \times \vec{F})(P)$,

and we get
$$\text{angular velocity} = \frac{1}{2} (\nabla \times \vec{F})(P) \cdot \frac{(\nabla \times \vec{F})(P)}{\|(\nabla \times \vec{F})(P)\|}$$

$$= \boxed{\frac{1}{2} \|(\nabla \times \vec{F})(P)\|}.$$

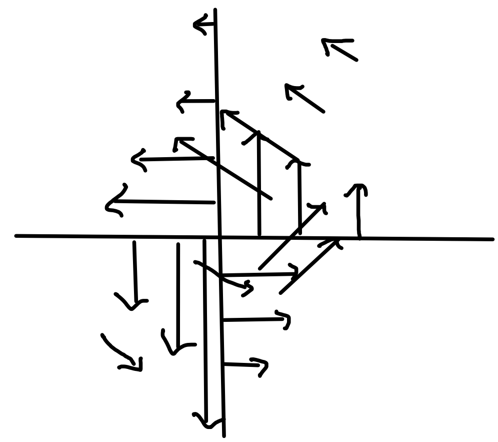
So $\text{curl}(\vec{F})(P)$ points in direction of greatest rotation, and amount of rotation is $\frac{1}{2}\|\text{curl}(\vec{F})(P)\|$.

Terminology Say $\vec{F}$ is irrotational if $\nabla \times \vec{F} = \vec{0}$.

Saw that for $\vec{F} = (-y, x, 1)$ get $\nabla \times \vec{F} = 2\vec{k}$.

Same for $\vec{F} = (-y, x, 0)$.

But for $\vec{F} = \frac{1}{x^2+y^2}(-y, x, 0)$, find $\nabla \times \vec{F} = \vec{0}$.



We already did this calculation (in Lecture 21)

since if $\vec{F} = (P, Q, 0)$ then

$$\nabla \times \vec{F} = \left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right).$$

Like a draining bathtube
