Lecture 4

Aug. 29 2011

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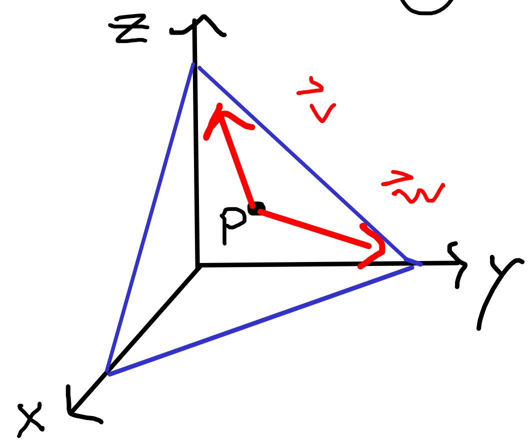


Last time: $\mathcal{P}$ plane.

P point on $\mathcal{P}$, vectors $\vec{v}$ & $\vec{w}$ parallel to $\mathcal{P}$. (not multiples of each other)

Then point Q on $\mathcal{P}$

$\iff$ can find scalars s & t with $\boxed{Q = P + s\vec{v} + t\vec{w}}$

Example $\mathcal{P}$ plane containing $\vec{i}, \vec{j}, \vec{k}$.

Take $P = (1,0,0)$

Choose $\vec{v} = \vec{j} - \vec{i} = (-1, 1, 0)$

$\vec{w} = \vec{k} - \vec{i} = (-1, 0, 1)$.

Is $Q = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$ on $\mathcal{P}$?

Want s, t so that

$$\frac{1}{2} = \underline{1} + s(-1) + t(-1)$$

$$\frac{1}{3} = 0 + s(1) + t(0)$$

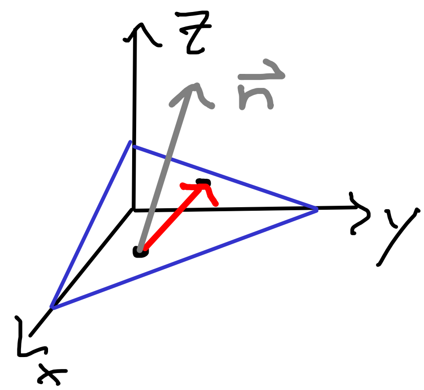
$$\frac{1}{6} = 0 + s(0) + t(1)$$

$\Rightarrow t = \frac{1}{6}, \; s = \frac{1}{3}$. <u>YES</u>

Ok, but not always best approach.

Suppose have vector $\vec{n}$ such that for any P & Q on $\mathcal{P}$

$$\vec{n} \perp \overrightarrow{PQ}.$$



$\vec{n}$ is a <u>normal vector</u> to the plane $\mathcal{P}$.

<u>Example</u> $\vec{i}$ is normal vector to y, z plane.

But $\vec{n} \perp \overrightarrow{PQ} \iff \vec{n} \cdot \overrightarrow{PQ} = 0$

If $\vec{u}$ & $\vec{v}$ are position vectors for P & Q, then $\overrightarrow{PQ} = \vec{v} - \vec{u}$, so $\vec{n} \cdot (\vec{v} - \vec{u}) = 0$, or

$$\boxed{\vec{n} \cdot \vec{v} = \vec{n} \cdot \vec{u}}$$

So given P & $\vec{n}$, have simple equation to determine if Q on $\mathcal{P}$.

<u>Example</u> Same plane $\mathcal{P}$ as above.

What is $\vec{n}$?

Want $\vec{n} \cdot \vec{v} = 0$ <u>and</u> $\vec{n} \cdot \vec{w} = 0$

Write $\vec{n} = (n_1, n_2, n_3)$.

$$0 = \vec{n}\cdot\vec{v} = -n_1 + n_2$$

$$0 = \vec{n}\cdot\vec{w} = -n_1 + n_3$$

$\Longrightarrow$ can choose $\vec{n} = (1,1,1)$.

Now Q on $\mathcal{P} \Longleftrightarrow \overrightarrow{PQ}\cdot(1,1,1) = 0$.
For $Q = (\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$, $\overrightarrow{PQ} = (-\frac{1}{2}, \frac{1}{3}, \frac{1}{6})$,
and $(-\frac{1}{2}, \frac{1}{3}, \frac{1}{6})\cdot(1,1,1) = -\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 0$.

Note: $Q = (x,y,z)$ on $\mathcal{P} \Longleftrightarrow$

$$\underbrace{(1,1,1)\cdot(x,y,z)}_{x+y+z} = \underbrace{(1,1,1)\cdot(1,0,0)}_{1}$$

We have recovered the equation for the plane!

Ok, how do we find $\vec{n}$ in general?
Suppose have $\vec{v}$ & $\vec{w}$. How do we find vector $\vec{n}$ orthogonal to both $\vec{v}$ & $\vec{w}$?

§ 12.4 The Cross Product

A second way to multiply vectors.

First, recall determinants of matrices.

For 2×2 matrices, this is

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

For 3×3, this is

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

Let $\vec{v} = (v_1, v_2, v_3)$, $\vec{w} = (w_1, w_2, w_3)$.

Define the <u>cross product</u>

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$$= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}\vec{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}\vec{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\vec{k}$$

<u>Properties</u>:

1. (<u>Anti</u> commutative) $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$
2. (Bilinear) a) $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$
   b) $(r\vec{v}) \times \vec{w} = r(\vec{v} \times \vec{w})$
3. (Orthogonality) $\vec{v} \perp \vec{v} \times \vec{w}$,
   $\vec{w} \perp \vec{v} \times \vec{w}$

Ex 1) $\vec{i} \times \vec{j} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0\vec{i} - 0\vec{j} + 1\vec{k}$

$= \vec{k}$

This follows Right Hand Rule.

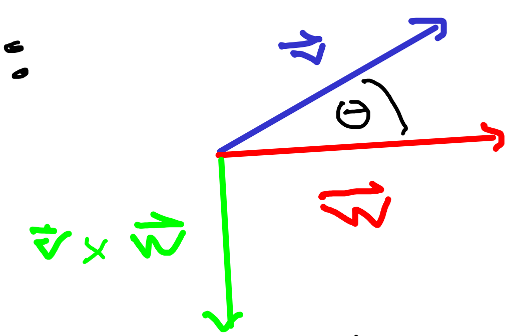
$\vec{j} \times \vec{i} = -\vec{k}$

2) $(-1,1,0) \times (-1,0,1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$

$= 1\vec{i} - (-1)\vec{j} + 1\vec{k} = \vec{i} + \vec{j} + k$

$= (1,1,1).$

Last key property:



$\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \, \|\vec{w}\| \sin\theta$

(we only allow $0 \le \theta \le \pi$, so $\sin\theta \ge 0$)

Reason: By calculation, can see

$\|\vec{v} \times \vec{w}\|^2 = \|\vec{v}\|^2 \|\vec{w}\|^2 - \underbrace{(\vec{v} \cdot \vec{w})}_{\|\vec{v}\| \, \|\vec{w}\| \cos\theta}{}^2$

Examples 1) saw $\vec{i} \times \vec{j} = \vec{k}$, and $\theta = 90°$

$1 = \|\vec{k}\| = \|\vec{i}\| \, \|\vec{j}\| \sin 90° = 1 \cdot 1 \cdot 1$

2) $(-1,1,0) \times (-1,0,1) = (1,1,1)$. $\theta = 60°$

$\sqrt{3} = \|(1,1,1)\| = \|(-1,1,0)\| \, \|(-1,0,1)\| \sin 60°$

$= \sqrt{2} \, \sqrt{2} \cdot \sqrt{3}/2$

<u>Note:</u> $\|\vec{v}\| \, \|\vec{w}\| \sin\theta$

$= \text{area of parallelogram},$

since $\text{area} = \text{base} \cdot \text{height}$

$= \|\vec{w}\| \cdot \left(\|\vec{v}\| \sin\theta\right)$