Tuesday, September 6 $\quad **\quad$ *Cross products and quadrics in* $\mathbb{R}^3$.

1. (a) Show that if $\mathbf{u}$ is any vector then $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.

   (b) If $\mathbf{u}$ and $\mathbf{v}$ are any two nonzero vectors such that $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, what can you say about the vectors $\mathbf{u}$ and $\mathbf{v}$?

   (c) Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors that are not parallel to each other. Show that the vector

   $$\mathbf{u} \times (\mathbf{u} \times \mathbf{v})$$

   can never be zero.

2. Suppose that $\mathbf{v} \cdot \mathbf{w} = 0$. Find an expression for $\|\mathbf{v} \times \mathbf{w}\|$ (in terms of $\mathbf{v}$ and $\mathbf{w}$).

3. (Volume of Prisms) Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors. Consider the prism (parallelepiped) with vertex at the origin and with sides given by the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. The formula for the volume of a prism, like that of a cylinder, is

   $$\text{volume} = \text{base} \cdot \text{height}$$

   (a) Consider the face containing $\mathbf{u}$ and $\mathbf{v}$ as the “base”, give the formula for the area of the base.

   (b) Find a formula for the height. This formula should be expressed in terms of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and the angle θ between $\mathbf{w}$ and the plane containing $\mathbf{u}$ and $\mathbf{v}$.

   (c) Put your answers together to arrive at a formula for the volume of the prism.

4. (A quadric surface in nonstandard form) Consider the surface described by the equation

   $$4x^2 - 4xy + 4y^2 - 10x + 2y - 2z + 9 = 0.$$

   (a) Introduce new variables $u = x + y$ and $v = x - y$. Solve for x and y in terms of u and v.

   (b) Substitute your answer above into the original equation to get a new equation in terms of u, v, and z.

   (c) Complete the square in both u and v to in order to arrive at a quadric equation in “standard form”.

   (d) What type of surface is this? If you don’t remember the classification of quadric surfaces, try drawing the cross-sections with the standard coordinate planes.