Tuesday, September 27 $\quad **\quad$ *Taylor series, the* 2^{nd} *derivative test, and changing coordinates.*

1. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function having partial derivatives of all orders. The **Taylor series** of f centered around $\mathbf{c} = (a, b)$ is a power series in x and y of the form

$$\begin{aligned} T(f, \mathbf{c}) &= f(\mathbf{c}) + \alpha_{1,0}(x-a) + \alpha_{0,1}(y-b) \\ &\quad + \alpha_{2,0}(x-a)^2 + \alpha_{1,1}(x-a)(y-b) + \alpha_{0,2}(y-b)^2 + \text{higher order terms} \end{aligned}$$

   (a) Assume that the Taylor series converges to f, so that
   $$f(x,y) = T(f,\mathbf{c})(x,y)$$
   (at least in a disk around $\mathbf{c}$). Take partial derivatives of both sides with respect to x to find the coefficient $\alpha_{1,0}$. Use $\frac{\partial}{\partial y}$ to find $\alpha_{0,1}$.

   (b) Use second order partial derivatives to find the coefficients $\alpha_{2,0}$, $\alpha_{1,1}$, and $\alpha_{0,2}$.

2. Consider $f(x,y) = 2\cos x - y^2 + e^{xy}$.

   (a) Show that $(0,0)$ is a critical point for f.

   (b) Calculate each of f_{xx}, f_{xy}, f_{yy} at $(0,0)$ and use this to write out the 2^{nd}-order Taylor approximation for f at $(0,0)$.

   (c) To make sure the next two problems go smoothly, check your answer to (b) with the instructor.

3. Let $g(x,y)$ be the approximation you obtained for $f(x,y)$ near $(0,0)$ in 1(b).

   (a) It's not clear from the formula whether g, and hence f, has a min, max, or a saddle at $(0,0)$. Test along several lines until you are convinced you've determined which type it is.

   (b) Check that you're right in (a) using the 2^{nd}-derivative test. The next problem will help explain why this test works.

4. Consider alternate coordinates on $\mathbb{R}^2$ where (u,v) corresponds to $u(1,1) + v(-1,1)$.

   (a) Sketch the u- and v-axes, and draw the points whose (u,v)-coordinates are: $(-1,2)$, $(1,1)$, $(1,-1)$.

   (b) Give the general formula for the (x,y)-coordinates of a point in terms of u and v. (Like $x = r\cos\theta$ and $y = r\sin\theta$ in polar coordinates.)

   (c) Use (b) to express g as a function of u and v, and expand and simplify the resulting expression.

   (d) Explain why your answer in 3(c) confirms your answer in 2.

   (e) Sketch a few level sets for g. What do the level sets of f look like near $(0,0)$?

It turns out that there is always a similar change of coordinates so that the Taylor series of a function f which has a critical point at $(0,0)$ looks like $f(u,v) \approx f(0,0) + au^2 + bv^2$.