Practice Final Exam for Math 241

1. Consider the points $A = (2, 0, 1)$ and $B = (4, 2, 5)$ in $\mathbb{R}^3$.

   (a) Find the point M which is halfway between A and B on the line segment L joining them. **(2 pts)**

   (b) Find the equation for the plane P consisting of all points that are equidistant from A and B. **(3 pts)**

2. Consider the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

   (a) Compute the following limit, if it exists. **(4 pts)**

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

   (b) Where on $\mathbb{R}^2$ is the function f continuous? **(1 pts)**

3. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = xy$.

   (a) Use Lagrange multipliers to find the global (absolute) max and min of f on the circle $x^2 + y^2 = 2$. **(6 pts)**

   (b) If they exist, find the global min and max of f on $D = \{x^2 + y^2 \leq 2\}$. **(2 pts)**

   (c) For each critical point in the interior of D you found in part (b), classify it as a local min, local max, or saddle. **(2 pt)**

   (d) If they exist, find the global min and max of f on $\mathbb{R}^2$. **(2 pts)**

4. A function $f\colon \mathbb{R}^2 \to \mathbb{R}$ takes on the values shown in the table at right.

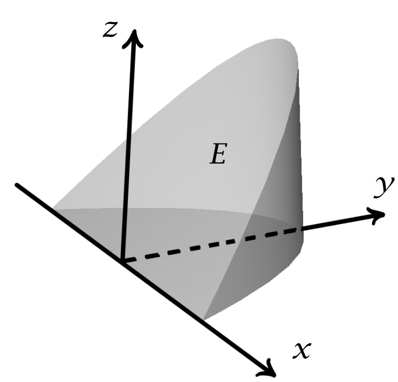
   (a) Estimate the partials $f_x(1,1)$ and $f_y(1,1)$. **(2 pts)**

   (b) Use your answer in (a) to approximate $f(1.1, 1.2)$. **(2 pts)**

   (c) Determine the sign of $f_{xy}(1,1)$: positive negative zero **(1 pt)**

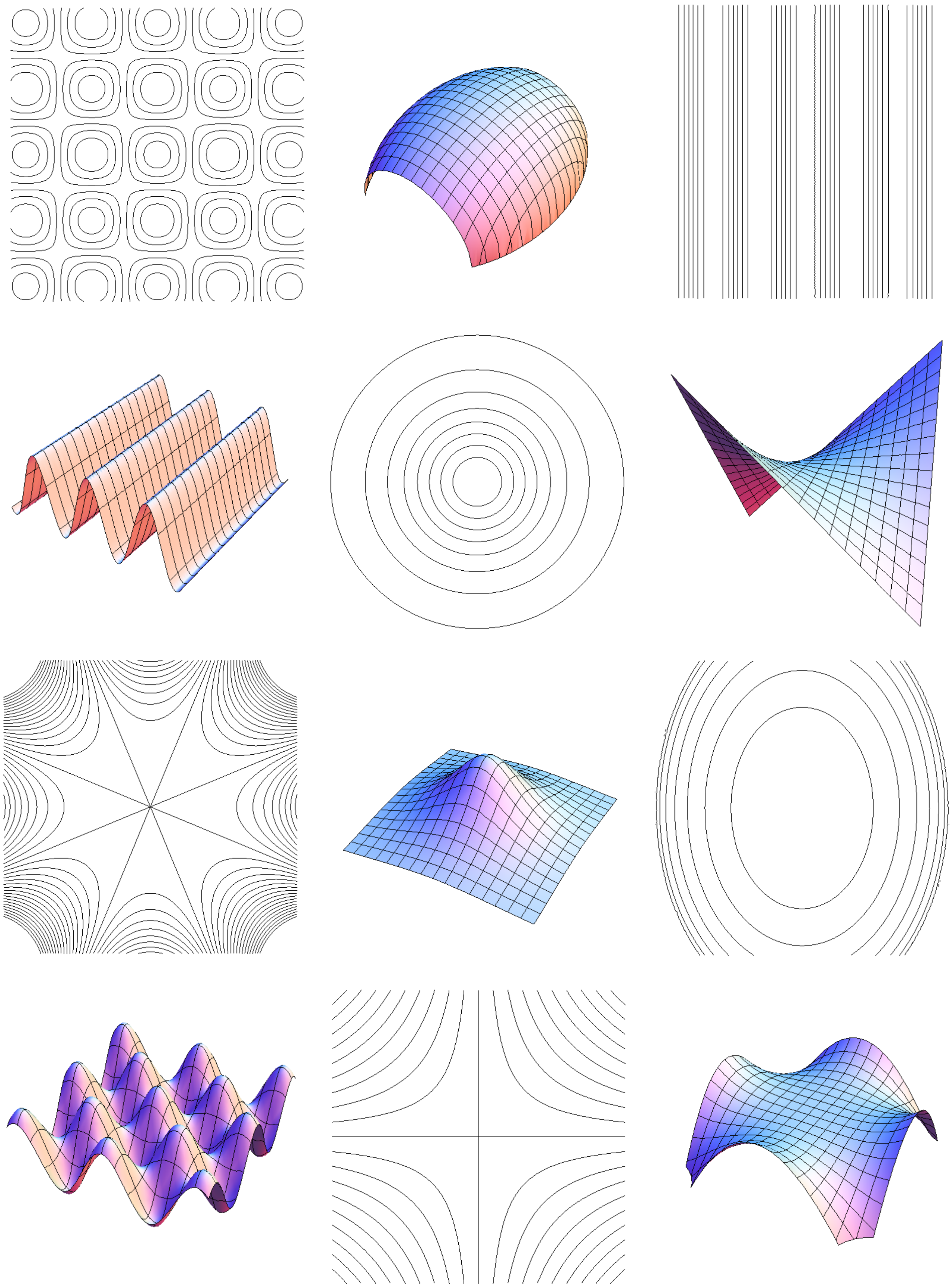
		x				
		0.2	0.6	1.0	1.4	1.8
	1.8	3.16	3.88	4.60	5.32	6.04
	1.4	2.68	3.24	3.80	4.36	4.92
y	1.0	2.20	2.60	3.00	3.40	3.80
	0.6	1.72	1.96	2.20	2.44	2.68
	0.2	1.24	1.32	1.40	1.48	1.56

5. Consider the region E shown at right, which is bounded by the xy-plane, the plane $z - y = 0$ and the surface $x^2 + y = 1$. Complete setup, but do not evaluate, a triple integral that computes the volume of E. **(6 pts)**



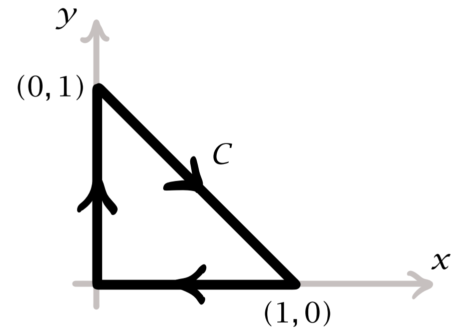
6. Match the following functions $\mathbb{R}^2 \to \mathbb{R}$ with their graphs and contour diagrams. Here each contour diagram consists of level sets $\{f(x,y) = c_i\}$ drawn for evenly spaced c_i. **(9 pts)**

(a) $\sqrt{8 - 2x^2 - y^2}$ (b) $\cos x$ (c) xy

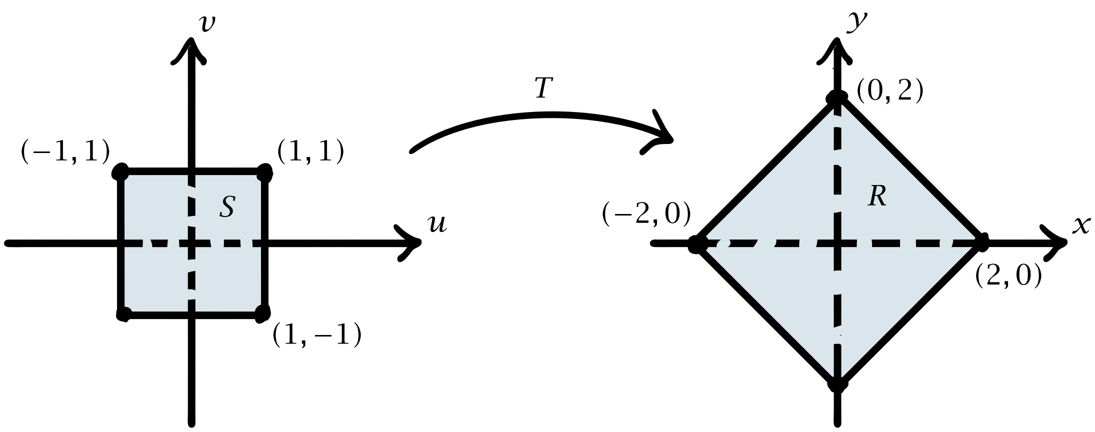


7. Consider the portion R of the cylinder $x^2 + y^2 \leq 2$ which lies in the positive octant and below the plane $z = 1$. Compute the total mass of R when it is composed of material of density $\rho = e^{x^2+y^2}$. **(7 pts)**

8. For the curve C in $\mathbb{R}^2$ shown and the vector field $\mathbf{F} = \left(\ln(\sin(x)), \cos(\sin(y)) + x\right)$ evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ using the method of your choice. **(5 pts)**



9. Let R be the region shown at right.



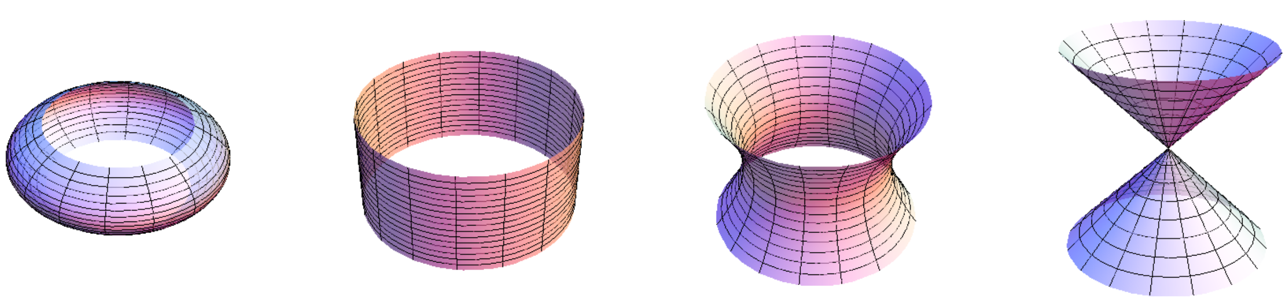
(a) Find a transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ taking $S = [-1,1] \times [-1,1]$ to R. **(4 pts)**

(b) Use your change of coordinates to evaluate $\int_R y^2 \, dA$ via an integral over S. **(6 pts)**

Emergency backup transformation: If you can't do (a), pretend you got the answer $T(u, v) = (uv, u + v)$ and do part (b) anyway.

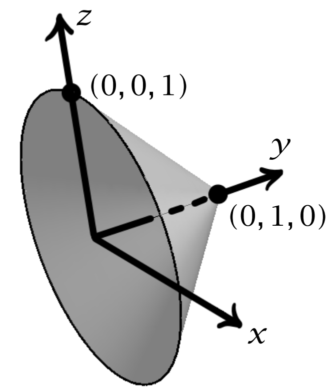
10. Consider the surface S which is parameterized by $\mathbf{r}(u, v) = \left(\sqrt{1+u^2}\cos v, \sqrt{1+u^2}\sin v, u\right)$ for $-1 \leq u \leq 1$ and $0 \leq v \leq 2\pi$.

(a) Circle the picture of S. **(2 pts)**



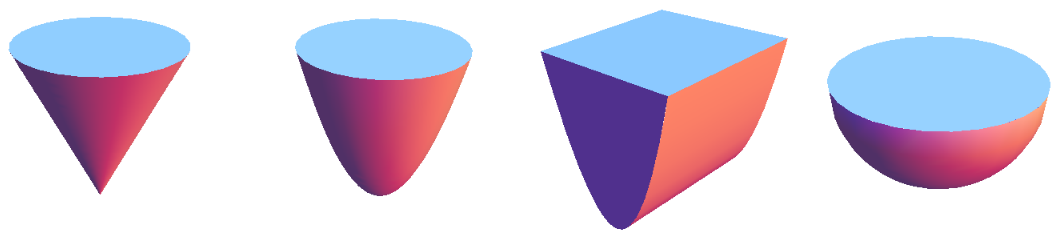
(b) Completely setup, but do not evaluate, an integral that computes the surface area of S. **(6 pts)**

11. For the cone S at right, give a parameterization $\mathbf{r}\colon D \to S$. Explicitly specify the domain D. **(5 pts)**



12. Consider the region R in $\mathbb{R}^3$ above the surface $x^2 + y^2 - z = 4$ and below the xy-plane. Also consider the vector field $\mathbf{F} = \langle 0, 0, z \rangle$.

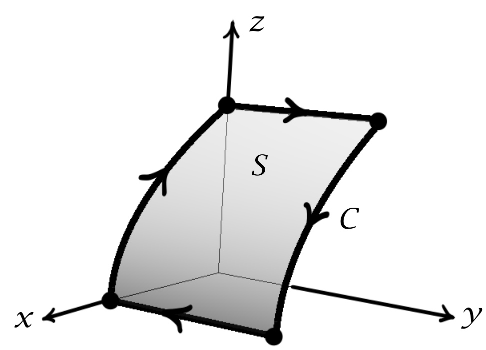
   (a) Circle the picture of R below. **(2 pts)**



   (b) Directly calculate the flux of $\mathbf{F}$ through the entire surface ∂R, with respect to the outward unit normals. **(10 pts)**

   (c) Use the Divergence Theorem and your answer in (b) to compute the volume of R. **(3 pts)**

13. Let C be the curve shown at right, which is the boundary of the portion of the surface $x + z^2 = 1$ in the positive octant where additionally $y \leq 1$.



   (a) Label the four corners of C with their (x, y, z)-coordinates. **(1 pt)**

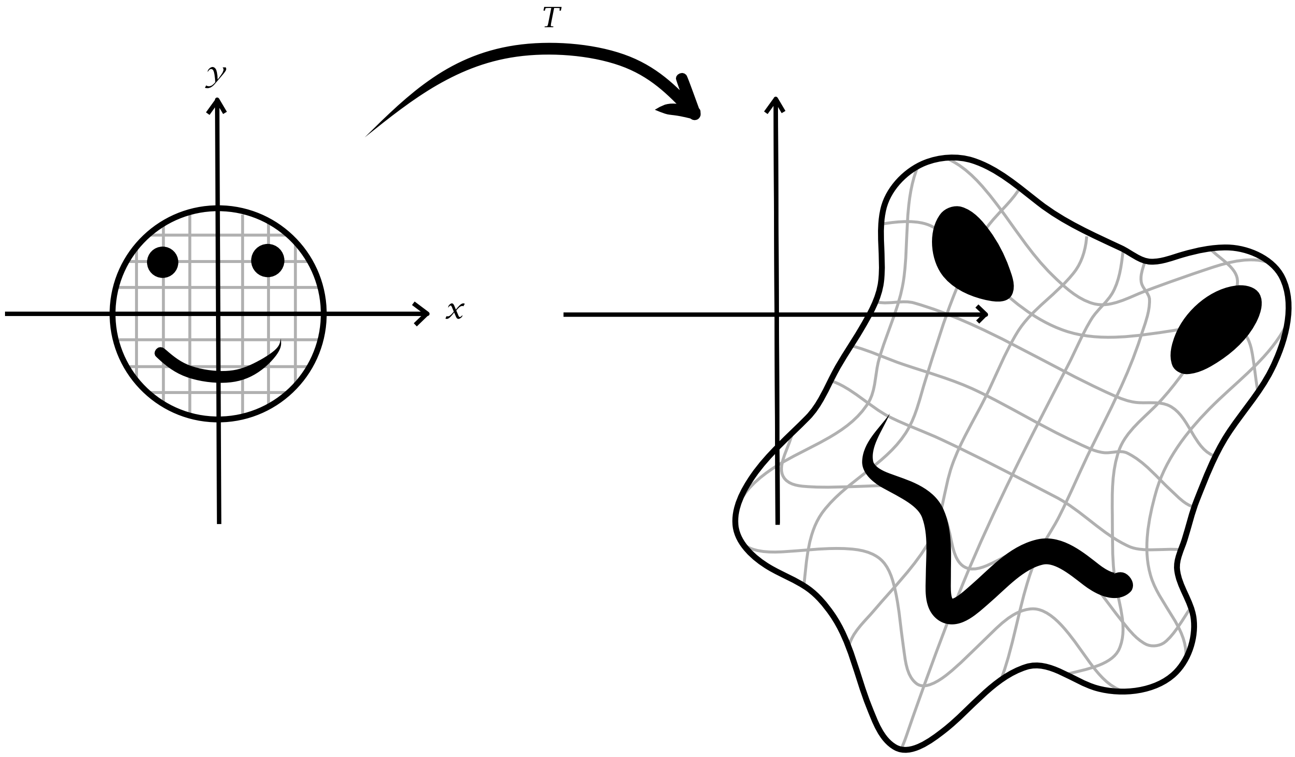
   (b) For $\mathbf{F} = \langle 0, xyz, xyz \rangle$, directly compute $\int_C \mathbf{F} \cdot d\mathbf{r}$. **(6 pts)**

   (c) Compute $\operatorname{curl} \mathbf{F}$. **(2 pts)**

   (d) Use Stokes' Theorem to compute the flux of $\operatorname{curl} \mathbf{F}$ through the surface S where the normals point out from the origin. **(3 pts)**

   (e) Give two distinct reasons why the vector field $\mathbf{F}$ is *not* conservative. **(2 pts)**

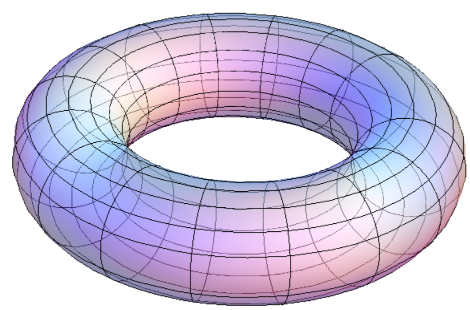
Extra Credit 1: Consider the transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ which distorts the plane as shown below:



(a) Draw in $T(0,0)$ on the right-hand part of the picture. **(1 pt)**

(b) Compute the Jacobian matrix of T at $(0,0)$, taking it as given that the entries of the matrix are integers. Hint: Tear off the bottom of this page to form a makeshift ruler. **(3 pts)**

Extra Credit 2: Consider the torus T shown below where the inner radius is 2 and the outer radius is 4, and hence the radius of tube itself is 1.



1. Compute the volume of T by computing the flux of some vector field $\mathbf{F}$. **(3 pts)**
2. Compute the volume of T via a 3-dimensional change of coordinates where your final integral is over a rectangular box. **(2 pts)**