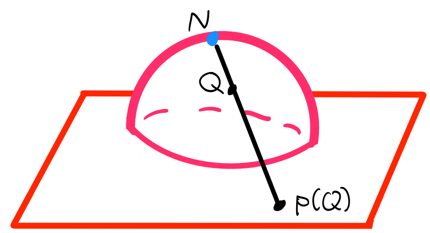
Group: ______________ ______________ ______________ Name: _______________

Math 351 - Elementary Topology
Friday, October 19 $**$ *Stereographic projection*

One way to visualize the two-sphere S^2 is as the plane $\mathbb{R}^2$ with a single point added in "at infinity". The homeomorphism that allows you to go back and forth is known as the *stereographic projection*. Let $N = (0, 0, 1)$ denote the north pole on the sphere.



1. Define a map $p : S^2 \setminus \{N\} \xrightarrow{\cong} \mathbb{R}^2$ as follows. For each point $Q = (x, y, z) \in S^2$, consider the straight line through N and Q. This line may be *parametrized* as
$$N + t(Q - N).$$
Find the unique point on this line for which $z = 0$ (solve for t). Define $p(Q)$ to be this point.

2. What are conditions on the point Q that determine whether $p(Q)$ will be inside the unit disc $B_1(\mathbf{0})$ or outside the disc?

3. Define a map $g : \mathbb{R}^2 \xrightarrow{\cong} S^2 \setminus \{N\}$ as follows. Given a point $P = (x, y) \in \mathbb{R}^2$, we may again consider the line through N and $Q = (x, y, 0)$ parametrized as before. Solve for the value of t that will put this point on the sphere S^2. (Remember that (x, y, z) is on the sphere if and only if $x^2 + y^2 + z^2 = 1$.)

4. Show that the maps p and g are homeomorphisms and that $g = p^{-1}$.

Write your answer(s) on the rest of this sheet (and back).

Solutions.

1. A point on the line has coordinates
$$(tx, ty, 1 - t + tz),$$
so the third coordinate will be zero when $t = \frac{1}{1-z}$. It follows that
$$p(Q) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

2. The point $p(Q)$ will be inside the disc when
$$\left(\frac{x}{1-z}\right)^2 + \left(\frac{y}{1-z}\right)^2 < 1.$$

The left-hand side is

$$\left(\frac{x}{1-z}\right)^2 + \left(\frac{y}{1-z}\right)^2 = \frac{x^2+y^2}{(1-z)^2} = \frac{1-z^2}{(1-z)^2} = \frac{1+z}{1-z}.$$

We have used the fact that $x^2+y^2+z^2=1$ in the second-to-last equality. The ratio $\frac{1+z}{1-z}$ is less than 1 precisely when $z<0$ and greater than 1 when $z>0$.
In other words, $p(Q)$ will be inside the disc when Q is on the southern hemisphere and outside the disc when Q is on the northern hemisphere.

3. We want to find a value of t that will put the point $(tx, ty, 1-t)$ on the sphere. This means solving for t in the equation

$$t^2x^2 + t^2y^2 + (1-t)^2 = 1.$$

Expanding the term on the left gives

$$t^2x^2 + t^2y^2 + 1 - 2t + t^2 = 1.$$

Subtracting 1 from both sides and factoring gives

$$t(tx^2 + ty^2 - 2 + t) = 0,$$

in other words $t = 0$ (the north pole) or $t = \frac{2}{x^2+y^2+1}$. It follows that we should define

$$g(x,y) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right).$$

4. To see these are homeomorphisms, observe first that they are both continuous. This is especially easy to see now that we know that for any space X, a map $f : X \longrightarrow \mathbb{R}^n$ is continuous if and only if each of the coordinate maps $f_i : X \longrightarrow \mathbb{R}$ are continuous. For instance, $p : S^2 \setminus \{N\} \longrightarrow \mathbb{R}^2$ is continuous since the two maps $\frac{x}{1-z}$ and $\frac{y}{1-z}$ are continuous (as long as $z \neq 1$).
Finally, to conclude that p and g are homeomorphisms, it remains to show that $pg = \mathrm{id}$ and $gp = \mathrm{id}$. The calculation is

$$pg(x,y) = p\left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1}\right).$$

Now

$$1 - \frac{x^2+y^2-1}{x^2+y^2+1} = \frac{2}{x^2+y^2+1},$$

so

$$pg(x,y) = \left(\frac{\frac{2x}{x^2+y^2+1}}{\frac{2}{x^2+y^2+1}}, \frac{\frac{2y}{x^2+y^2+1}}{\frac{2}{x^2+y^2+1}}\right) = (x,y).$$

Similarly,

$$gp(x,y,z) = g\left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

Now

$$\left(\frac{x}{1-z}\right)^2+\left(\frac{y}{1-z}\right)^2+1=\frac{x^2+y^2+1-2z+z^2}{(1-z)^2}=\frac{2-2z}{(1-z)^2}=\frac{2}{1-z}$$

and

$$\left(\frac{x}{1-z}\right)^2+\left(\frac{y}{1-z}\right)^2-1=\frac{x^2+y^2-1+2z-z^2}{(1-z)^2}=\frac{2z(1-z)}{(1-z)^2}=\frac{2z}{1-z}.$$

So

$$g\left(\frac{x}{1-z},\frac{y}{1-z}\right)=\left(\frac{\cancel{2\frac{x}{1-z}}}{\cancel{\frac{2}{1-z}}},\frac{\cancel{2\frac{y}{1-z}}}{\cancel{\frac{2}{1-z}}},\frac{\cancel{\frac{2z}{1-z}}}{\cancel{\frac{2}{1-z}}}\right)=(x,y,z).$$

We have shown that p is a bijection with inverse given by g.