Math 551 - Topology I
Homework 5
Fall 2013

1. (a) Assume that a continuous map $q : X \longrightarrow Y$ has a "section", meaning a continuous map $s : Y \longrightarrow X$ such that $q \circ s = \mathrm{id}_Y$. Show that q is necessarily a quotient.
   (b) Let $A \subseteq X$ be a subspace. A map $r : X \longrightarrow A$ is called a **retraction** if $r \circ \iota_A = \mathrm{id}_A$, where $\iota_A : A \longrightarrow X$ is the subspace inclusion. Show that a retraction is necessarily a quotient map.

2. Show that if X is discrete, then $X \times Y \cong \coprod_X Y$.

3. Find a continuous surjection $q : D^n \longrightarrow S^n$ that takes the boundary to the north pole and the origin to the south pole, inducing a continuous bijection $D^n/\partial D^n \cong S^n$. We will see later that such a map is automatically a homeomorphism.

   (Hint: One way to do this is to find appropriate $\mathbb{R}$-valued functions c and d on D^n and define
   $$q(\mathbf{x}) = (c(\mathbf{x}) \cdot \mathbf{x}, d(\mathbf{x})).$$
   Start by defining $d(\mathbf{x})$ and use the defining property of S^n.)

4. A **based space** is simply a space X with a chosen basepoint $x_0 \in X$. We say that a map $f : X \longrightarrow Y$ is based (or basepoint-preserving) if $f(x_0) = y_0$, where x_0 and y_0 are the chosen basepoints of X and Y, respectively.

   Let (X, x_0) and (Y, y_0) be based spaces. We define their **wedge sum** to be
   $$X \vee Y = X \coprod Y / \sim$$
   where $\iota_X(x_0) \sim \iota_Y(y_0)$.

   (a) Show that the wedge sum satisfies the universal property of the coproduct for based spaces. That is, if $f : (X, x_0) \longrightarrow (Z, z_0)$ and $g : (Y, y_0) \longrightarrow (Z, z_0)$ are based maps, there is a unique continuous map $h : (X \vee Y, \overline{x_0}) \longrightarrow (Z, z_0)$ which makes the appropriate diagram commute.
   (b) Show that if X and Y are Hausdorff based spaces, then so is $X \vee Y$.

5. Given based spaces (X, x_0) and (Y, y_0), there is a natural axes inclusion $X \vee Y \hookrightarrow X \times Y$. Define the **smash product** of X and Y to be
   $$X \wedge Y = (X \times Y)/(X \vee Y).$$
   Show that there is a homeomorphism $S^1 \wedge S^1 \cong S^2$ or that more generally $S^n \wedge S^1 \cong S^{n+1}$. (Hint: Feel free to assume the existence of a homeomorphism $D^n \cong I^n$ that takes the boundary to the boundary.)