Math 551 - Topology I
Homework 7
Fall 2013

1. (Cantor set) Let $A_0 = I = [0,1]$. Define $A_1 = A_0 \setminus (\frac{1}{3}, \frac{2}{3})$. Similarly, define A_2 by removing the middle thirds of the intervals in A_1:

$$A_2 = A_1 \setminus \left(\left(\frac{1}{9}, \frac{2}{9} \right) \cup \left(\frac{7}{9}, \frac{8}{9} \right) \right).$$

In general, given A_n constructed in this way, we define A_{n+1} by removing the middle thirds of all intervals in A_n. Define the Cantor set to be

$$C = \bigcap_n A_n \subseteq [0,1].$$

   (a) Show that C is compact (without using part (d)).

   (b) Show that any compact, locally connected space has finitely many components. Conclude that C is not locally connected.

   (c) Show that C is totally disconnected (every connected component is a singleton).

   (d) Let $D = \{0,2\}$ with the discrete topology. Show that $C \cong \prod_n D$. (Hint: instead of binary expansions, think about ternary expansions of numbers in $[0,1]$.)

2. Let X be Hausdorff, and suppose that $C, D \subseteq X$ are disjoint compact subsets. Show that there are disjoint open sets $U, V \subseteq X$ with $C \subseteq U$ and $D \subseteq V$.

3. (Stereographic Projection) Let $N = (0, \dots, 0, 1) \in S^n$ be the North Pole. Define a homeomorphism $S^n \setminus \{N\} \cong \mathbb{R}^n$ as follows. For each $x \neq N \in S^n$, consider the ray starting at N and passing through x. This meets the equatorial hyperplane (defined by $x_{n+1} = 0$) in a point, which we call $p(x)$.

   (a) Determine a formula for p and show that it gives a homeomorphism.

   (b) Conclude that the one-point compactification of $\mathbb{R}^n$ is S^n.

4. Show that if Z is locally compact Hausdorff and $q : X \longrightarrow Y$ is a quotient, then

$$q \times \mathrm{id}_Z : X \times Z \longrightarrow Y \times Z$$

is a quotient.