Math 551 - Topology I
Homework 9.5
Fall 2013

1. A map $f : X \longrightarrow Y$ is said to be **perfect** if it is a closed surjection and for every $y \in Y$, the fiber $f^{-1}(y)$ is compact.

   (a) Show that if U is open in X and $f^{-1}(y) \subset U$, then y has a neighborhood V in Y such that $f^{-1}(V) \subset U$.

   (b) Show that if $f : X \longrightarrow Y$ is perfect and X is second-countable, so is Y. (Hint: Let $\mathcal{B}$ be a countable basis for X. For each finite subset J of $\mathcal{B}$, let $B_J = \bigcup_{B \in J} B$, and let

   $$U_J = \bigcup_{\substack{W \subseteq Y \text{open} \\ f^{-1}(W) \subseteq B_J}} f^{-1}(W).$$

   You may also find part (a) useful.)

   (c) Let Y be a CW complex. We showed in class that Y is a quotient $q : \coprod_i D_i^{n_i} \twoheadrightarrow Y$ of all of its closed cells. Show that this is a perfect map.

   (d) Conclude that if Y has countably many cells, then it is paracompact.

   (In fact, it is true that any CW complex is paracompact, but the general proof is more difficult.)

2. ($\star$) In this problem, you will give a CW structure for $S^1 \times S^2$.

   (a) Show that $\partial I^3 = (\partial I^1 \times I^2) \cup (I^1 \times \partial I^2)$ in I^3.

   (b) Consider S^1 and S^2 with their minimal CW structures, each having two cells. Build a 3-dimensional CW complex X as follows. Start with a single 0-cell, and attach both a 1-cell and a 2-cell to this point. The result is $S^1 \vee S^2$. Now attach a 3-cell, using (a) to help you define the attaching map.

   (c) Find a homeomorphism $X \xrightarrow{\cong} S^1 \times S^2$. (Since X is compact and $S^1 \times S^2$ is Hausdorff, it suffices to find a continuous bijection.)