Math 551 - Topology I
Homework 9
Fall 2013

1. Fill in the details to show that $\mathbb{RP}^n$ is an n-manifold.

2. Recall that for any $P \in S^1$, the punctured circle $U_P = S^1 \setminus \{P\}$ is homeomorphic to $\mathbb{R}^1$.

   (a) Pick two points $P, Q \in S^1$ and cover S^1 by U_P and U_Q. Find a partition of unity subordinate to this cover.

   (b) Following the embedding theorem from class, this gives an embedding $E : S^1 \hookrightarrow \mathbb{R}^4$. Describe the map $S^1 \longrightarrow \mathbb{R}^2$ obtained by using the first two coordinates of this embedding. Following notation from the proof of the embedding theorem, this would be the map
   $$(f_1\varphi_1, f_2\varphi_2) : S^1 \longrightarrow \mathbb{R}^2.$$
   In particular, you should show that this is *not* an embedding.

3. Show that if Y is a metric space, A is any space, and $\mathcal{C}(A, Y)$ is given the topology coming from the uniform metric, then the evaluation map $\mathrm{ev} : \mathcal{C}(A, Y) \times A \longrightarrow Y$ is continuous.

4. Show that if X and Y are any spaces and A is discrete, then
   $$\mathcal{C}(X \times A, Y) \cong \mathcal{C}(X, Y^A)$$
   if Y^A is given the product topology. (Hint: You may find problem 2 from HW 5 helpful.)

5. Suppose that Y is Hausdorff and A is any space. Show that $\mathrm{Map}(A, Y)$ is Hausdorff.

6. Let X and Y be any spaces. Let $g : Y \longrightarrow Z$ be a continuous map to a third space Z. Show that composition with g defines a continuous function
   $$G : \mathrm{Map}(X, Y) \longrightarrow \mathrm{Map}(X, Z), \qquad G(f) = g \circ f.$$

7. ($\star$) Let Z be locally compact Hausdorff and $q : X \longrightarrow Y$ a quotient map. This describes an alternate proof that $q \times \mathrm{id}_Z$ is a quotient map.

   (a) Let Q be the quotient space defined by the map $\langle q \rangle = q \times \mathrm{id}_Z$, and let $d : X \times Z \longrightarrow Q$ be the quotient map. Show that there is a continuous bijection $f : Q \longrightarrow Y \times Z$ with $f \circ d = \langle q \rangle$.

   (b) Define $g = f^{-1}$. Let
   $$G : Y \longrightarrow \mathrm{Map}(Z, Q) \qquad \text{and} \qquad D : X \longrightarrow \mathrm{Map}(Z, Q)$$
   be adjoint to g and d, respectively. Show that $D = G \circ q$.

   (c) Show that Q is continuous. Conclude that G and therefore g are also continuous. This shows that f is a homeomorphism, so that $\langle q \rangle$ is a quotient.