26. Mon, Oct. 25

Definition 26.1. We say that a space is **locally compact** if every $x \in X$ has a compact neighborhood (recall that we do not require neighborhoods to be open).

This looks different from our other "local" notions. To get a statement in the form we expect, we introduce more terminology $A \subseteq X$ is **precompact** if $\overline{A}$ is compact.

Proposition 26.2. *Let X be Hausdorff. TFAE*

(1) *X is locally compact*
(2) *every $x \in X$ has a precompact neighborhood*
(3) *X has a basis of precompact open sets*

Proof. It is clear that (3) $\Rightarrow$ (2) $\Rightarrow$ (1) without the the Hausdorff assumption, so we show that (1) $\Rightarrow$ (3). Suppose X is locally compact and Hausdorff. Let V be open in X and let $x \in V$. We want a precompact open neighborhood of x in V. Since X is locally compact, we have a compact neighborhood K of x, and since X is Hausdorff, K must be closed. Since V and K are both neighborhoods of x, so is $V \cap K$. Thus let $x \in U \subseteq V \cap K$. Then $\overline{U} \subseteq K$ since K is closed, and $\overline{U}$ is compact since it is a closed subset of a compact set. ■

In contrast to the local connectivity properties, it is clear that any compact space is locally compact. But this is certainly a generalization of compactness, since *any interval* in $\mathbb{R}$ is locally compact.

Example 26.3. A standard example of a space that is not locally compact is $\mathbb{Q} \subseteq \mathbb{R}$. We show that 0 does not have any compact neighborhoods . Let V be any neighborhood of 0. Then it must contain $(-\pi/n, \pi/n)$ for some n. Now

$$\mathcal{U} = \left\{ \left(-\pi/n, \left(\frac{k}{k+1} \right) \pi/n \right) \right\} \cup \left\{ V \cap (\pi/n, \infty), V \cap (-\infty, -\pi/n) \right\}$$

is an open cover of V with no finite subcover.

Locally compact Hausdorff spaces are a very nice class of spaces (almost as good as compact Hausdorff). In fact, any such space is close to a compact Hausdorff space.

Definition 26.4. A **compactification** of a noncompact space X is an embedding $i : X \hookrightarrow Y$, where Y is compact and $i(X)$ is dense.

Example 26.5. The open interval $(0, 1)$ is not compact, but $(0, 1) \hookrightarrow [0, 1]$ is a compactification. Note that the exponential map $\exp : (0, 1) \longrightarrow S^1$ also gives a (different) compactification.

There is often a smallest compactification, given by the following construction.

Definition 26.6. Let X be a space and define $\widehat{X} = X \cup \{\infty\}$, where $U \subseteq \widehat{X}$ is open if either

- $U \subseteq X$ and U is open in X or
- $\infty \in U$ and $\widehat{X} \setminus U \subseteq X$ is compact.

Proposition 26.7. *Suppose that X is Hausdorff and noncompact. Then $\widehat{X}$ is a compactification. If X is locally compact, then $\widehat{X}$ is Hausdorff.*

Proof. We first show that $\widehat{X}$ is a space! It is clear that both $\emptyset$ and $\widehat{X}$ are open.

Suppose that U_1 and U_2 are open. We wish to show that $U_1 \cap U_2$ is open.

- If neither open set contains ∞, this follows since X is a space.

- If $\infty \in U_1$ but $\infty \notin U_2$, then $K_1 = X \setminus U_1$ is compact. Since X is Hausdorff, K_1 is closed in X. Thus $X \setminus K_1 = U_1 \setminus \{\infty\}$ is open in X, and it follows that $U_1 \cap U_2 = (U_1 \setminus \{\infty\}) \cap U_2$ is open.
- If $\infty \in U_1 \cap U_2$, then $K_1 = X \setminus U_1$ and $K_2 = X \setminus U_2$ are compact. It follows that $K_1 \cup K_2$ is compact, so that $U_1 \cap U_2 = X \setminus (K_1 \cup K_2)$ is open.
- Suppose we have a collection U_i of open sets. If none contain ∞, then neither does $\bigcup_i U_i$, and the union is open in X. If $\infty \in U_j$ for some j, then $\infty \in \bigcup_i U_i$ and
$$\widehat{X} \setminus \bigcup_i U_i = \bigcap_i (\widehat{X} \setminus U_i) = \bigcap_i (X \setminus U_i)$$
is a closed subset of the compact set $X \setminus U_j$, so it must be compact.

Next, we show that $\iota : X \longrightarrow \widehat{X}$ is an embedding. Continuity of ι again uses that compact subsets of X are closed. That ι is open follows immediately from the definition of $\widehat{X}$.

27. Wed, Oct. 30

Recall from last time that we were in the midst of the proof of the following result:

Proposition 27.1. *Suppose that X is Hausdorff and noncompact. Then $\widehat{X}$ is a compactification. If X is locally compact, then $\widehat{X}$ is Hausdorff.*

We already showed that $\widehat{X}$ is a space and that $\iota : X \longrightarrow \widehat{X}$ is an embedding.

To see that $\iota(X)$ is dense in $\widehat{X}$, it suffices to see that $\{\infty\}$ is not open. But this follows from the definition of $\widehat{X}$, since X is not compact.

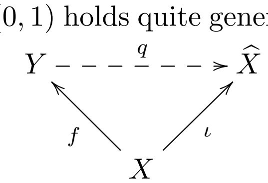
Finally, we show that $\widehat{X}$ is compact. Let $\mathcal{U}$ be an open cover. Then some $U \in \mathcal{U}$ must contain ∞. The remaining elements of $\mathcal{U}$ must cover $X \setminus U$, which is compact. It follows that we can cover $X \setminus U$ using only finitely many elements, so $\mathcal{U}$ has a finite subcover.

Now suppose that X is locally compact. Let x_1 and x_2 in $\widehat{X}$. If neither is ∞, then we have disjoint neighborhoods in X, and these are still disjoint neighborhoods in $\widehat{X}$. If $x_2 = \infty$, let $x_1 \in U \subseteq K$, where U is open and K is compact. Then U and $V = \widehat{X} \setminus K$ are the desired disjoint neighborhoods
. ■

Remark 27.2. Why did we define local compactness in a different way from local (path)-connectedness? We could have defined locally connected to mean that every point has a connected neighborhood , which follows from the actual definition. But then we would not have that locally connected is equivalent to having a basis of connected open sets. On the other hand, we could try the $x \in K \subseteq U$ version of locally compact, but of course we don't want to allow $K = \{x\}$, so the next thing to require is $x \in V \subseteq U$, where V is precompact. As we showed in Prop 26.2, this is equivalent to our definition of locally compact in the presence of the Hausdorff condition. Without the Hausdorff condition, compactness does not behave quite how we expect.

Next, we show that the situation we observed for compactifications of $(0, 1)$ holds quite generally.

Proposition 27.3. *Let X be locally compact Hausdorff and let $f : X \longrightarrow Y$ be a compactification. Then there is a (unique) quotient map $q : Y \longrightarrow \widehat{X}$ such that $q \circ f = \iota$.*



We will need:

Lemma 27.4. *Let X be locally compact Hausdorff and $f : X \longrightarrow Y$ a compactification. Then f is open.*

Proof. Since f is an emebedding, we can pretend that $X \subseteq Y$ and that f is simply the inclusion. We wish to show that X is open in Y. Thus let $x \in X$. Let U be a precompact neighborhood of x. Thus $K = \mathrm{cl}_X(U)$ is compact[2] and so must be closed in Y (and X) since Y is Hausdorff. By the definition of the subspace topology, we must have $U = V \cap X$ for some open $V \subseteq Y$. Then V is a neighborhood of x in Y, and

$$V = V \cap \mathrm{cl}_Y(X) \subseteq \mathrm{cl}_Y(V \cap X) = K \subseteq X.$$

■

Proof of Prop. 27.3. We define

$$q(y) = \begin{cases} \iota(x) & \text{if } y = f(x) \\ \infty & \text{if } y \notin f(X). \end{cases}$$

To see that q is continuous, let $U \subseteq \widehat{X}$ be open. If $\infty \notin U$, then $q^{-1}(U) = f(\iota^{-1}(U))$ is open by the lemma. If $\infty \in U$, then $K = \widehat{X} \setminus U$ is compact and thus closed. We have $q^{-1}(K) = f(\iota^{-1}(K))$ is compact and closed in Y, so it follows that $q^{-1}(U) = Y \setminus q^{-1}(K)$ is open.

Note that q is automatically a quotient map since it is a closed continuous surjection (it is closed because Y is compact and $\widehat{X}$ is Hausdorff). Note also that q is unique because $\widehat{X}$ is Hausdorff and q is already specified on the dense subset $f(X) \subseteq Y$. ■

In categorical language, the one-point compactification is a "terminal object" in the category of compactifications of X. One might then ask if there is a compactification on "the other side", meaning a compactification Y such that any other compactification is a quotient of Y? We will come back to this point.

28. Fri, Nov. 1

Let's turn back to compactness in metric spaces. We already saw (Theorem 25.5) that compactness in a metric space is equivalent to the statement that every sequence has a convergent subsequence. We also saw that the compact subsets of the metric space $\mathbb{R}^n$ are the closed and bounded ones. Do we have an analogue of the second statement for an arbitrary metric space X? First, note that closed and bounded is not enough in general to guarantee compactness, as any infinite discrete metric space shows.

Definition 28.1. We say that a metric space X is **totally bounded** if, for every $\epsilon > 0$, there is a finite covering of X by ϵ-balls.

It is clear that compact implies totally bounded because, for any fixed $\epsilon > 0$, the B_ϵ give an open covering. This suffices to handle the discrete metric case, as a discrete metric space is totally bounded $\iff$ it is finite $\iff$ it is compact. However, closed and totally bounded is still not enough, as $[0, 1] \cap \mathbb{Q}$ is closed and totally bounded (either in $\mathbb{Q}$ or in itself) but not compact. Recall that a metric space is **complete** if every Cauchy sequence converges in X.

Theorem 28.2. *Let X be metric. Then X is compact $\iff$ X is complete and totally bounded.*

[2] We will need to distinguish between closures in X and closures in Y, so we use the notation $\mathrm{cl}_X(A)$ for closure rather than our usual $\overline{A}$.

Proof. (⇒) We have already mentioned why compactness implies totally bounded. Let (x_n) be a Cauchy sequence in X. Then, since X is sequentially compact, a subsequence of (x_n) converges. But if $x_{n_k} \to x$, then we must also have $x_n \to x$ since x_n is Cauchy (prove this)! It follows that X is complete.

(⇐) Suppose now that X is complete and totally bounded. We show that X is sequentially compact. Let (x_n) be any sequence in X. Since X is complete, it suffices to show that (x_n) has a subsequence that is Cauchy.

For each n, we have a finite covering of X by k_n balls of radius $1/n$. Start with $n = 1$. One of these balls must contain infinitely many x_n's and so a subsequence of (x_n). Now cover X by finitely many balls of radius $1/2$. Again, one of these contains a subsequence of the previous subsequence. We continue in this way ad infinitum. We obtain the desired Cauchy subsequence as follows. First, pick x_{n_1} to be in our original subsequence (in the chosen ball of radius 1). Then pick x_{n_2} to be in the subsubsequence in our chosen ball of radius $1/2$ (and pick it such that $n_2 > n_1$. After (many, many) choices, we get a subsequence of x_n such that $\{x_{n_k}\}_{k\geq m}$ is contained in a ball of radius $1/m$. It follows that x_{n_k} is Cauchy. ■

Theorem 28.3 (Baire "Category" Theorem)**.** *Let X be either locally compact Hausdorff or complete metric. Then every countable collection of dense open sets has dense intersection.*

Proof. See either Munkres, Theorem 48.2 or Lee, Theorem 4.68. ■

One standard application of this result is as follows.

Proposition 28.4. *Let X be either locally compact Hausdorff or complete metric. Suppose that no singletons in X are open. Then X is uncountable.*

Proof. Let $\{x_n\}$ be a countable subset of X. We show that $C = \{x_n\}$ is a proper subset of X. For each n, let $U_n = X \setminus \{x_n\}$. Then each U_n is open (X is Hausdorff) and dense ($\{x_n\}$ is not open). Then $\bigcap_n U_n$ is dense and therefore nonempty. But the intersection is precisely the complement of C. ■

Another famous application is the existence of a continuous but *nowhere* differentiable function (see §49 of Munkres).

Remark 28.5. Note that if we apply the one-point compactification to a (locally compact) metric space X, there is no natural metric to put on X, so one might ask for a good notion of compactification for metric spaces. Given the result above, this should be related to the idea of a completion of a metric space. See HW8.

Proposition 28.6. *A space X is Hausdorff and locally compact if and only if it is homeomorphic to an open subset of a compact Hausdorff space Y.*

Proof. (⇒). We saw that X is open in the compact Hausdorff space $Y = \widehat{X}$.

(⇐) As a subspace of a Hausdorff space, it is clear that X is Hausdorff. It remains to show that every point has a compact neighborhood (in X). Write $Y_\infty = Y \setminus X$. This is closed in Y and therefore compact. By Problem 2 from HW7, we can find disjoint open sets $x \in U$ and $Y_\infty \subseteq V$ in Y. Then $K = Y \setminus V$ is the desired compact neighborhood of x in X. ■

Corollary 28.7. *If X and Y are locally compact Hausdorff, then so is $X \times Y$.*

Corollary 28.8. *Any open or closed subset of a locally compact Hausdorff space is locally compact Hausdorff.*