Another related concept is that of paracompactness. This is especially important in the theory of manifolds and vector bundles. We make a couple of preliminary definitions first.

Definition 29.1. If $\mathcal{U}$ and $\mathcal{W}$ are collections of subsets of X, we say that $\mathcal{W}$ is a **refinement** of $\mathcal{U}$ if every $W \in \mathcal{W}$ is a subset of some $U \in \mathcal{U}$.

Definition 29.2. An open cover $\mathcal{U}$ of X is said to be **locally finite** if every $x \in X$ has a neighborhood meeting only finitely many elements of the cover.

For example, the covering $\{(n, n+2) \mid n \in \mathbb{Z}\}$ of $\mathbb{R}$ is locally finite.

Definition 29.3. A space X is said to be **paracompact** if every open cover has a locally finite refinement.

From the definition, it is clear that compact implies paracompact. But this really is a generalization, as the next example shows.

Proposition 29.4. *The space $\mathbb{R}$ is paracompact.*

Proof. Let $\mathcal{U}$ be an open cover of $\mathbb{R}$. For each $n \geq 0$, let $A_n = \pm[n, n+1]$ and $W_n = \pm(n - \frac{1}{2}, n + \frac{3}{2})$. Then $A_n \subset W_n$, A_n is compact and W_n is open. (We take $W_0 = (-\frac{3}{2}, \frac{3}{2})$.) Fix an n. For each $x \in A_n$, pick a $U_x \in \mathcal{U}$ with $x \in U_x$, and let $V_x = U_x \cap W_n$. The V_x's give an open cover of A_n, and so there is a finite collection $\mathcal{V}_n$ of V_x's that will cover A_n. Then $\mathcal{V} = \bigcup_n \mathcal{V}_n$ gives a locally finite refinement of $\mathcal{U}$. (Note that only W_{n-1}, W_n, and W_{n+1} meet the subset A_n). ■

This argument adapts easily to show that $\mathbb{R}^n$ is paracompact. In fact, something more general is true. First we make a definition.

Definition 29.5. A space X is said to be **second countable** if it has a countable basis.

Lemma 29.6. *Any open cover of a second countable space has a countable subcover.*

Proof. Given a countable basis $\mathcal{B}$ and an open cover $\mathcal{U}$, we first replace the basis by the countable subset $\mathcal{B}'$ consisting of those basis elements that are entirely contained in some open set from the cover (this is a basis too, but we don't need that). For each $B \in \mathcal{B}'$, pick some $U_B \in \mathcal{U}$ containing B, and let $\mathcal{U}' \subseteq \mathcal{U}$ be the (countable) collection of such U_B. It only remains to observe that $\mathcal{U}'$ is still a cover, because

$$\bigcup_{\mathcal{U}'} U_B \supset \bigcup_{\mathcal{B}'} B = X.$$

■

Proposition 29.7. *Every second countable, locally compact Hausdorff space is paracompact.*

The proof strategy is the same. The assumptions give you a cover (basis) by precompact sets and thus a countable cover by precompact sets. You use this to manufacture a countable collection of compact sets A_n and open sets W_n that cover X as above. The rest of the proof is the same.

Note that of the assumptions in the proposition, locally compact and Hausdorff are both *local* properties, whereas second countable is a global property. As we will see, paracompactness (and therefore the assumptions in this proposition) is enough to guarantee the existence of some nice functions on a space.

We finally turn to the so-called "separation axioms".

Definition 29.8. A space X is said to be

- T_0 if given two distinct points x and y, there is a neighborhood of one not containing the other

- T_1 if given two distinct points x and y, there is a neighborhood of x not containing y and vice versa (points are closed)
- T_2 (**Hausdorff**) if any two distinct points x and y have disjoint neighborhoods
- T_3 (**regular**) if *points are closed and* given a closed subset A and $x \notin A$, there are disjoint open sets U and V with $A \subseteq U$ and $x \in V$
- T_4 (**normal**) if *points are closed and* given closed disjoint subsets A and B, there are disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$.

Note that $T_4 \implies T_3 \implies T_2 \implies T_1 \implies T_0$. But beware that in some literature, the "points are closed" clause is not included in the definition of regular or normal. Without that, we would not be able to deduce T_2 from T_3 or T_4.

Proposition 29.9. *Any compact Hausdorff space is normal.*

Proof. This was homework problem 7.2. ∎

Proposition 29.10. *Like the Hausdorff condition, regularity is inherited by subspaces but not by images or quotients.*

Example 29.11. We will see that $\mathbb{R}$ is normal. But recall the quotient map $q : \mathbb{R} \longrightarrow \{-1, 0, 1\}$ which sends any number to its sign. This quotient is not Hausdorff and therefore not regular or normal.

30. Wed, Nov. 6

Proposition 30.1. *Let $X_i \neq \emptyset$ for all i. Then $\prod_i X_i$ is regular if and only if each X_i is regular.*

Proof. ($\Rightarrow$) This works just like the Hausdorff proof. We can embed each X_i into the product and use that a subspace of a regular space is regular.

($\Leftarrow$)

Lemma 30.2. *Suppose that X is T_1. Then X is regular if and only if for every point x and neighborhood U of x, there is a neighborhood V of x with $x \in V \subseteq \overline{V} \subseteq U$.*

We write $X = \prod_i X_i$ for simplicity. Assume each X_i is regular and nonempty. Note that each X_i is therefore Hausdorff, and so we know that X is Hausdorff and thus T_1. Now let $x \in X$ and let U be an open neighborhood of x. We may as well assume that U is a basic open set. Thus $U = \prod_i U_i$, where $U_i = X_i$ for all but finitely many i. For each i, we have $x_i \in U_i$. Since X_i is regular, we can find a neighborhood V_i of x_i with $x_i \in V_i \subseteq \overline{V_i} \subset U_i$. In the case $U_i = X_i$, we take $V_i = X_i$ too.

Now $(x_i) \in \prod_i V_i \subseteq \overline{\prod_i V_i} = \prod_i \overline{V_i} \subseteq \prod_i U_i = U$. This shows that X is regular. ∎

It is simple to show that a subspace of a regular space is again regular, but the obvious argument there does not translate to subspaces of normal spaces. It is easy to show that a *closed* subspace of a normal space is again normal. We will see an example that shows arbitrary subspaces of normal may not be normal.

Example 30.3. The lower limit topology $\mathbb{R}_{\ell\ell}$ is normal, but $\mathbb{R}_{\ell\ell} \times \mathbb{R}_{\ell\ell}$ is not normal. This gives an example of a regular space that is not normal and also shows that products of normal spaces need not be normal. (See Munkres, example 31.3)

Another important class of normal spaces is the collection of metric spaces.

Proposition 30.4. *If X is metric, then it is normal.*

Proof. Let X be metric and let $A, B \subseteq X$ be closed and dijsoint. For every $a \in A$, let $\epsilon_a > 0$ be a number such that $B_{\epsilon_a}(a)$ does not meet B (using that B is closed). Let

$$U_A = \bigcup_{a \in A} B_{\epsilon_a/2}(a).$$

Similarly, we let

$$U_B = \bigcup_{b \in B} B_{\epsilon_b/2}(b).$$

It only remains to show that U_A and U_B must be disjoint. Let $x \in B_{\epsilon_a/2}(a) \subseteq U_A$ and pick any $b \in B$. We have

$$d(a, x) < \frac{1}{2}\epsilon_a < \frac{1}{2}d(a, b)$$

and thus

$$d(x, b) \geq d(a, b) - d(a, x) > d(a, b) - \frac{1}{2}d(a, b) = \frac{1}{2}d(a, b) > \frac{1}{2}\epsilon_b.$$

It follows that $U_A \cap U_B = \emptyset$. ■

Ok, so we've seen a few examples. So what, why should we care about normal spaces? Look back at the definition for T_2, T_3, T_4. In each case, we need to find certain open sets U and V. How would one do this in general? In a metric space, we would build these up by taking unions of balls. In an arbitrary space, we might use a basis. But another way of getting open sets is by pulling back open sets under a continuous map.

Theorem 30.5 (Urysohn's Lemma)**.** *Let X be normal and let A and B be disjoint closed subsets. Then there exists a continuous function $f : X \longrightarrow [0, 1]$ such that $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(B)$.*

Sketch of proof. Define $U_1 = X \setminus B$. Since X is normal, we can find an open U_0 with $A \subseteq U_0 \subseteq \overline{U_0} \subseteq U_1$. By induction on the rational numbers $r \in \mathbb{Q} \cap (0, 1)$, we can find for each r an open set U_r with $\overline{U_r} \subset U_s$ if $r < s$. We also define $U_r = X$ for $r > 1$. Then define

$$f(x) = \inf\{r \in \mathbb{Q} \cap [0, 1] \mid x \in U_r\}.$$

It follows that $A \subseteq f^{-1}(0)$ and $B \subseteq f^{-1}(B)$ as desired. It remains to show that f is continuous.

It suffices to show that the preimage under f of the subbasis elements $(-\infty, a)$ and (a, ∞) are open. We have

$$f^{-1}(-\infty, a) = \bigcup_{\substack{r \in \mathbb{Q} \\ r < a}} U_r, \qquad \text{and} \qquad f^{-1}(a, \infty) = X \setminus \bigcap_{\substack{r \in Q \\ r > a}} \overline{U_r}$$

For more details, see either [Lee, Thm 4.82] or [Munkres, Thm 33.1]. ■

Note that Urysohn's Lemma becomes an if and only if statement if we either drop the T_1-condition from normal or if we explicitly include singletons as possible replacements for A and B.

31. Fri, Nov. 8

Last time, we saw that a space is normal if and only if any two closed sets can be separated by a continuous function (modulo the T_1 condition). Here is another important application of normal spaces.

Theorem 31.1 (Tietze extension theorem)**.** *Suppose X is normal and $A \subseteq X$ is closed. Then any continuous function $f : A \longrightarrow [0, 1]$ can be extended to a continuous function $\tilde{f} : X \longrightarrow [0, 1]$.*

Again, this becomes an if and only if if we drop the T_1-condition from normal.

It is also easy to see that the result fails if we drop the hypothesis that A be closed. Consider $X = S^1$ and A is the complement of a point. Then we know that $A \cong (0,1)$, but this homeomorphism cannot extend to a map $S^1 \twoheadrightarrow (0,1)$.

Sketch of proof. It is more convenient for the purpose of the proof to work with the interval $[-1,1]$ rather than $[0,1]$. Thus suppose $f : A \longrightarrow [-1,1]$ is continuous. Then $A_1 = f^{-1}([-1,-1/3])$ and $A_2 = f^{-1}([1/3,1])$ are closed, disjoint subsets of A and therefore also of X. Since X is normal, we have a Urysohn function $g_1 : X \longrightarrow [-1/3,1/3]$ which separates A_1 and A_2. It is simple to check that $|f(a) - g_1(a)| < 2/3$ for all $a \in A$. In other words, we have a map

$$f' = f - g_1 : A \longrightarrow [-2/3, 2/3].$$

Define $A_1' = f'^{-1}([-2/3,-2/9])$ and $A_2' = f'^{-1}([2/9,2/3])$. We get a Urysohn function $g_2 : X \longrightarrow [-1/9,1/9]$ which separates A_1' and A_2'. Then the difference $f'' = f - g_1 - g_2$ maps to $[-2/9,2/9]$. We continue in this way, and in the end, we get a sequence of functions (g_n) defined on X, and we define $g = \sum_n g_n$. By construction, this agrees with f on A (the difference will be less than $2/3^n$ for all n). The work remains in showing that the series defining g converges (compare to a geometric series) and that the resulting g is continuous. See [Munkres, Thm 35.1] for more details. ∎

You might wonder if there is a version of Urysohn's lemma for regular spaces: that is, is it true that if a space is regular and A is a closed subset not containing a point x, then some continuous function separates x from A? It turns out that the answer is not quite. If you try to modify the proof of Urysohn's lemma for the regular case, you quickly get stuck. You get the open set U_0, but at the next stage, there is no reason that the U_r's must exist.

Definition 31.2. We say that X is **completely regular** (or $T_{3\frac{1}{2}}$) if it is T_1 and given a closed set A and a point x not in A, there exists a continuous f such that $f(x) = 0$ and $f(a) = 1$ for all $a \in A$.

We have $T_4 \implies T_{3\frac{1}{2}} \implies T_3$, but neither implication is an if and only if.

Theorem 31.3 (Stone-Čech compactification)**.** *Suppose X is completely regular. There exists a "universal" compactification $\iota : X \longrightarrow Y$ of X, such that if $j : X \longrightarrow Z$ is any map to a compact Hausdorff space (for example a compactification), there is a unique ~~quotient~~ map $q : Y \longrightarrow Z$ with $q \circ \iota = j$.*

$$\begin{tikzcd} & Y \arrow[dr, dashed, "q"] & \\ X \arrow[ur, hook, "\iota"] \arrow[rr, "j"'] & & Z \end{tikzcd}$$

Proof. Given the space X, let $\mathcal{F} = \{ \text{ cts } f : X \longrightarrow [0,1]\}$. Define

$$\iota : X \longrightarrow [0,1]^{\mathcal{F}}$$

by $\iota(x)_f = f(x)$. This is continuous because each coordinate function is given by some $f \in \mathcal{F}$. The infinite cube is compact Hausdorff, and we let $Y = \overline{\iota(X)}$. It remains to show that ι is an embedding and also to demonstrate the universal property.

First, ι is injective since X is completely regular: given distinct points x and y in X, there is a Urysohn function separating x and y, so $\iota(x) \neq \iota(y)$.

Now suppose that $U \subseteq X$ is open. We wish to show that $\iota(U)$ is open in $\iota(X)$. Pick $x_0 \in U$. Since X is completely regular, we have a function $g : X \longrightarrow [0,1]$ with $g(x_0) = 0$ and $g \equiv 1$ outside of U. Let

$$B = \{\iota(x) \in \iota(X) \mid g(x) \neq 1\}.$$

Certainly $\iota(x_0) \in B$. Also, this is open since it can be rewritten as $p_g^{-1}([0,1))$. Finally, $B \subset \iota_U)$ since if $\iota(x) \in B$, then $g(x) \neq 1$. But $g \equiv 1$ outside of U, so x must be in U.

For the universal property, suppose that $j : X \longrightarrow Z$ is a map to a compact Hausdorff space. Then Z is also completely regular, and the argument above shows that it embeds inside some large cube $[0,1]^K$. For each $k \in K$, we thus get a coordinate map $i_k : X \longrightarrow [0,1]$, and it is clear how to extend this to get a map $q_k : Y \longrightarrow [0,1]$: just take q_k to be the projection map p_{i_k} onto the factor labelled by the map i_k. Piecing these together gives a map $q : Y \longrightarrow [0,1]^K$, but it restricts to the map i on the subset X. Since i has image in the closed subset Z, it follows that $q(Y) \subseteq Z$ since q is continuous and $\iota(X)$ is dense in Y. Note that q is the *unique* extension of j to Y since Z is Hausdorff and $\iota(X)$ is dense in Y. ■