35. Mon, Nov. 18

Definition 35.1. A (topological) n**-manifold** M is a Hausdorff, second-countable space such that each point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

Example 35.2.

(1) $\mathbb{R}^n$ and any open subset is obviously an n-manifold

(2) S^1 is a 1-manifold. More generally, S^n is an n-manifold. Indeed, we have shown that if you remove a point from S^n, the resulting space is homeomorphic to $\mathbb{R}^n$.

(3) T^n, the n-torus, is an n-manifold. In general, if M is an m-manifold and N is an n-manifold, then $M \times N$ is an $(m+n)$-manifold.

(4) $\mathbb{RP}^n$ is an n-manifold. There is a standard covering of $\mathbb{RP}^n$ by open sets as follows. Recall that $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^\times$. For each $1 \le i \le n+1$, let $V_i \subseteq \mathbb{R}^{n+1} \setminus \{0\}$ be the complement of the hyperplane $x_i = 0$. This is an open, saturated set, and so its image $U_i = V_i/\mathbb{R}^\times \subseteq \mathbb{RP}^n$ is open. The V_i's cover $\mathbb{R}^{n+1} \setminus \{0\}$, so the U_i's cover $\mathbb{RP}^n$. We leave the rest of the details as an exercise.

(5) $\mathbb{CP}^n$ is a $2n$-manifold. This is similar to the description given above.

(6) $O(n)$ is a $\frac{n(n-1)}{2}$-manifold. Since it is also a topological group, this makes it a *Lie group*. The standard way to see that this is a manifold is to realize the orthogonal group as the preimage of the identity matrix under the transformation $M_n(R) \longrightarrow M_n(R)$ that sends A to A^TA. This map lands in the subspace $S_n(R)$ of symmetric $n \times n$ matrices. This space can be identified with $\mathbb{R}^{n(n+1)/2}$.

Now the $n \times n$ identity matrix is an element of S_n, and an important result in differential topology (Sard's theorem) that says that if a certain derivative map is surjective, then the preimage of the submanifold $\{I_n\}$ will be a submanifold of $M_n(\mathbb{R})$. in this case, the relevant derivative is the matrix of partial derivatives of $A \mapsto A^TA$, writen in a suitable basis.

(7) $\mathrm{Gr}_{k,n}(\mathbb{R})$ is a $k(n-k)$-manifold.

Here are some nonexamples

Example 35.3.

(1) The union of the coordinate axes in $\mathbb{R}^2$. Every point has a neighborhood like $\mathbb{R}^1$ except for the origin.
(2) A discrete uncountable set is not second countable.
(3) A 0-manifold is discrete, so $\mathbb{Q}$ is not a 0-manifold.
(4) Glue together two copies of $\mathbb{R}$ by identifying any nonzero x in one copy with the point x in the other. This is second-countable and looks locally like $\mathbb{R}^1$, but it is not Hausdorff.

36. Wed, Nov. 20

Note that since open subsets of $\mathbb{R}^n$ are locally compact, it follows from Prop 29.7 that any n-manifold is paracompact.

Theorem 36.1. *Any manifold M^n admits an embedding into some Euclidean space $\mathbb{R}^N$.*

Proof. The theorem is true as stated, but we only prove it in the case of a compact manifold. Note that in this case, since M is compact and $\mathbb{R}^N$ is Hausdorff, it is enough to find a continuous injection of M into some $\mathbb{R}^N$.

Since M is a manifold, it has an open cover by sets that are homeomorphic to $\mathbb{R}^n$. Since it is compact, there is a finite subcover $\{U_1, \dots, U_k\}$. Since M is paracompact, there is a partition of unity $\{\varphi_1, \dots, \varphi_k\}$ subordinate to this cover. For each i, let $f_i : U_i \xrightarrow{\cong} \mathbb{R}^n$ be a homeomorphism.

We can then piece these together as follows: for each $i = 1, \dots, k$, define $g_i : M \longrightarrow \mathbb{R}^n$ by

$$g_i(x) = \begin{cases} f_i(x)\varphi_i(x) & x \in U_i \\ \mathbf{0} & x \in X \setminus \operatorname{supp}(\varphi_i) \end{cases}.$$

Note that g_i is continuous by the glueing lemma, since $\operatorname{supp}(\varphi_i)$ is closed. Then the k functions g_i together give a continuous function $g : M \longrightarrow \mathbb{R}^{nk}$. Now suppose that $g(x) = g(x')$ for some $x, x' \in M$. Unfortunately, this need not be injective, since if $f_i(x) = \mathbf{0}$ and x does not lie in any other U_j, it follows that $g(x) = \mathbf{0}$. Since there can be more than one such x, we cannot conclude that g is injective.

One way to fix this would be to stick on the functions φ_i, in order to separate out points lying in different U_i's. Define $G = (g_1, \dots, g_k, \varphi_1, \dots, \varphi_k) : M \longrightarrow \mathbb{R}^{nk+k}$. But now G is injective, since if $G(x) = G(x')$ and we pick i so that $\varphi_i(x) = \varphi_i(x') > 0$, then this means that x and x' both lie in U_i. But then $g_i(x) = g_i(x')$ so $f_i(x) = f_i(x')$. Since f_i is a homeomorphism, it follows that $x = x'$. ■

In fact, one can do better. Munkres shows (Cor. 50.8) that every compact n-manifold embeds inside $\mathbb{R}^{2n+1}$.

The last main topic from the introductory part of the course on metric spaces is the idea of a function space. Given any two spaces A and Y, we will want to be able to define a topology on the set of continuous functions $A \longrightarrow Y$ in a sensible way. We already know one topology on Y^A, namely the product topology. But this does not use the topology on A at all.

Let's forget about topology for a second. A function $h : X \times A \longrightarrow Y$ between sets is equivalent to a function

$$\Psi(h) : X \longrightarrow Y^A.$$

Given h, the map $\Psi(h)$ is defined by $\big(\Psi(h)(x)\big)(a) = h(x, a)$. Conversely, given $\Psi(h)$, the function h can be recovered by the same formula.

Let's play the same game in topology. What we want to say is that a continuous map $h : X \times A \longrightarrow Y$ is the same as a continuous map $X \longrightarrow \operatorname{Map}(A, Y)$, for some appropriate *space* of maps $\operatorname{Map}(A, Y)$. Let's start by seeing why the product topology does *not* have this property.

We write $C(X, Z)$ for the *set* of continuous maps $X \longrightarrow Z$. It is not difficult to check that the set-theoretic construction from above does give a function

$$C(X \times A, Y) \longrightarrow C(X, Y^A),$$

where for the moment Y^A denotes the set of continuous functions $A \longrightarrow Y$ given the product topology. But this function is not surjective.

Example 36.2. Take $A = [0, 1]$, $Y = \mathbb{R}$, and $X = Y^A = \mathbb{R}^{[0,1]}$. We can consider the identity map $\mathbb{R}^{[0,1]} \longrightarrow \mathbb{R}^{[0,1]}$. We would like this to correspond to a continuous map $\mathbb{R}^{[0,1]} \times [0, 1] \longrightarrow \mathbb{R}$. We see that, ignoring the topology, this function must be the evaluation function $ev : (g, x) \mapsto g(x)$. But this is not continuous.

To see this consider $ev^{-1}((0, 1))$. The point $(\mathrm{id}, 1/2)$ lies in this preimage, but we claim that no neighborhood of this point is contained in the preimage. In fact, we claim no basic neighborhood $U \times (a, b)$ lies in the preimage. For such a U must consist of functions that are close to $\mathrm{id} : [0, 1] \longrightarrow \mathbb{R}$ at *finitely many points* $c_1, \dots, c_n$. So given any such U and any interval $(a, b) = (1/2 - \epsilon, 1/2 + \epsilon)$, pick any point $d \in (a, b)$ that is distinct from the c_i. It is simple to construct a continuous function $g : [0, 1] \longrightarrow \mathbb{R}$ such that (1) $g(c_i) = c_i$ for each i and (2) $g(d) = 2$. Then $(g, d) \in U \times (a, b)$ but $(g, d) \notin ev^{-1}((0, 1))$ since $ev(g, d) = g(d) = 2$.

37. Fri, Nov. 22

Last time, we were trying to find a topology $\mathrm{Map}(X,Y)$ on the set $\mathcal{C}(X,Y)$ of continuous functions $X \longrightarrow Y$ which would provide a bijection

$$\mathcal{C}(X \times A, Y) \cong \mathcal{C}(X, \mathrm{Map}(A,Y)).$$

This problem will be (almost) solved by the compact-open topology.

The **compact-open** topology on the set $\mathcal{C}(A,Y)$ has a sub-basis given by

$$S(K,U) = \{f : A \longrightarrow Y \mid f(K) \subseteq U\},$$

where K is compact and $U \subseteq Y$ is open.

Theorem 37.1. *Suppose that A is locally compact Hausdorff. Then a function $f : X \times A \longrightarrow Y$ is continuous if and only if the induced function $g = \Psi(f) : X \longrightarrow \mathcal{C}(A,Y)$ is continuous.*

Proof. ($\Rightarrow$) This direction does not need that A is locally compact. Before we give the proof, we should note why $\Psi(f)(x) : A \longrightarrow Y$ is continuous. This map is the composite $A \xrightarrow{\iota_x} X \times A \xrightarrow{f} Y$ and therefore continuous.

We now wish to show that $g = \Psi(f)$ is continuous. Let $S(K,U)$ be a sub-basis element in $\mathcal{C}(A,Y)$. We wish to show that $g^{-1}(S(K,U))$ is open in X. Let $g(x) = f(x,-) \in S(K,U)$. Since f is continuous, the preimage $f^{-1}(U) \subseteq X \times A$ is open. Furthermore, $\{x\} \times K \subseteq f^{-1}(U)$. We wish to use the Tube Lemma, so we restrict from $X \times A$ to $X \times K$. By the Tube Lemma, we can find a basic neighborhood V of x such that $V \times K \subseteq (X \times K) \cap f^{-1}(U)$. It follows that $g(V) \subseteq S(K,U)$, so that V is a neighborhood of x in $g^{-1}(S(K,U))$.

($\Leftarrow$) Suppose that g is continuous. Note that we can write f as the composition

$$X \times A \xrightarrow{g \times \mathrm{id}} \mathcal{C}(A,Y) \times A \xrightarrow{ev} Y,$$

so it is enough to show that ev is continuous.

Lemma 37.2. *The map $ev : C(A,Y) \times A \longrightarrow Y$ is continuous if A is locally compact Hausdorff.*

Proof. Let $U \subseteq Y$ be open and take a point (f,a) in $ev^{-1}(U)$. This means that $f(a) \in U$. Since A is locally compact Hausdorff, we can find a compact neighborhood K of a contained in $f^{-1}(U)$ (this is open since f is continuous). It follows that $S(K,U)$ is a neighborhood of f in $\mathcal{C}(A,Y)$, so that $S(K,U) \times K$ is a neighborhood of (f,a) in $ev^{-1}(U)$. ∎

∎

Even better, we have

Theorem 37.3. *Let X and A be locally compact Hausdorff. Then the above maps give homeomorphisms*

$$\mathrm{Map}(X \times A, Y) \cong \mathrm{Map}(X, \mathrm{Map}(A,Y)).$$

It is fairly simple to construct a continuous map in either direction, using Theorem 37.1. You should convince yourself that the two maps produced are in fact inverse to each other.

In practice, it's a bit annoying to keep track of these extra hypotheses at all times, especially since not all constructions will preserve these properties. It turns out that there is a "convenient" category of spaces, where everything works nicely.

Definition 37.4. A space A is **compactly generated** if a subset $B \subseteq A$ is closed if and only if for every map $u : K \longrightarrow A$, where K is compact Hausdorff, then $u^{-1}(B) \subseteq K$ is closed.

We say that the topology of A is determined (or generated) by compact subsets. Examples of compactly generated spaces include locally compact spaces and first countable spaces.

Definition 37.5. A space X is **weak Hausdorff** if the image of every $u : K \longrightarrow X$ is closed in X.

There is a way to turn any space into a weak Hausdorff compactly generated space. In that land, <u>everything</u> works well! For the most part, whenever an algebraic topologist says "space", they really mean a compactly generated weak Hausdorff space. Next semester, we will always implicity be working with spaces that are CGWH.