38. Mon, Nov. 25

Last week, we showed that the compact-open topology on a mapping space $\mathrm{Map}(A, Y)$ has the nice property that we in fact get a homeomorphism

$$\mathrm{Map}(X \times A, Y) \cong \mathrm{Map}(X, \mathrm{Map}(A, Y))$$

under mild hypotheses on A and X. Before getting to the compact-open topology, we saw why the product topology would not do (too coarse). There was the suggestion to also try the box topology, but here is an example to show why that also would not have worked.

Consider the case $X = Y = \mathbb{R}$ and $A = \mathbb{N}$, all with their usual topologies. Let $f : \mathbb{R} \times \mathbb{N} \longrightarrow \mathbb{R}$ be the projection. Then this corresponds to the function $\mathbb{R} \longrightarrow \mathbb{R}^N$ which sends any number to the constant sequence at that number. Another name for this function is the diagonal, and we saw when we introduced the box topology that it is not continuous.

Looking back to the initial discussion of metric spaces, there we introduced the uniform topology on a mapping space.

Theorem 38.1 (Munkres, 46.7 or Willard, 43.6). *Let Y be a metric space. Then on the set $\mathcal{C}(A, Y)$ of continuous functions $A \longrightarrow Y$, the compact-open topology is intermediate between the uniform topology and the product topology. Furthermore, the compact-open topology agrees with the uniform topology if A is compact.*

The main point is to show (Munkres, Theorem 46.8) that the compact-open topology can be described by basis elements

$$B_K(f, \epsilon) = \{g : A \longrightarrow Y \mid \sup_K d(f(x), g(x)) < \epsilon\}.$$

To see that this satisfies the intersection property for a basis, suppose that

$$g \in B_{K_1}(f_1, \epsilon_1) \cap B_{K_2}(f_2, \epsilon_2)$$

Write $m_i = \sup_{K_i} d(f_i(x), g(x))$ and $\delta_i = \epsilon_i - m_i$. Then the triangle inequality gives

$$B_{K_i}(g, \delta_i) \subseteq B_{K_i}(f_i, \epsilon_i).$$

It follows that, setting $\delta = \min\{\delta_1, \delta_2\}$

$$g \in B_{K_1 \cup K_2}(g, \delta) \subseteq B_{K_1}(f_1, \epsilon_1) \cap B_{K_2}(f_2, \epsilon_2).$$

In the setting of metric spaces, the compact-open topology is known as the topology of compact convergence, as convergence of functions corresponds to (uniform) convergence on compact subsets.

For fun, here is one of the first results towards the theory of C^*-algebras (pronounced C-star).

Theorem 38.2. *Let X be compact Hausdorff and denote by $C(X)$ the space $\mathrm{Map}(X, \mathbb{R})$ of real-valued functions on X. Then the map*

$$\Lambda : X \longrightarrow \widehat{C(X)} = \{\lambda : C(X) \longrightarrow \mathbb{R} \mid \lambda \textit{ is a continuous } \mathbb{R}\textit{-algebra map}\}$$

defined by

$$\Lambda(x) = \mathrm{ev}_x$$

is a homeomorphism if $\widehat{C(X)} \subseteq \prod_{C(X)} \mathbb{R}$ is equipped with the product topology.

In this case, the product topology coincides with a topology of interest in analysis known as the weak-$*$ topology.

Proof. Since we have given $\widehat{C(X)}$ the product topology, it is simple to verify that Λ is continuous. Note that since X is compact and $\widehat{C(X)}$ is Hausdorff, it remains only to show that Λ is a bijection.

Suppose that $\Lambda(x) = \Lambda(x')$. Since X is compact Hausdorff, for any two distinct points there is a continuous function taking different values at those points. The fact that $\Lambda(x) = \Lambda(x')$ says that no such function exists for x and x', so we must have $x = x'$.

Now let $\lambda \in \widehat{C(X)}$. We wish to show that $\lambda = \mathrm{ev}_x$ for some x.

The main step is to show that there exists $x \in X$ such that if $\lambda(f) = 0$ for some f, then $f(x) = 0$. Suppose not. Then for every $x \in X$, there exists a function f_x with $\lambda(f_x) = 0$ but $f_x(x) \neq 0$. For each x, let $U_x = f_x^{-1}(\mathbb{R} \setminus \{0\})$. Then the collection $\{U_x\}$ covers X since $x \in U_x$. As X is compact, there is a finite subcover $\{U_{x_1}, \dots, U_{x_n}\}$. Now define

$$g = f_{x_1}^2 + \cdots + f_{x_n}^2,$$

and note that $g(x) > 0$ for all x. This is because $f_{x_i} \neq 0$ on U_{x_i} and the U_{x_i} cover X. Since g is nonzero, it follows that $1/g$ is also continuous on X. But now

$$1 = \lambda(g \cdot 1/g) = \lambda(g) \cdot \lambda(1/g),$$

which implies that $\lambda(g) \neq 0$. But λ is an algebra homomorphism, so

$$\lambda(g) = \sum_i \lambda(f_{x_i})^2 > 0,$$

which is a contradiction.

This now establishes that there must be an $x \in X$ such that if $\lambda(f) = 0$ then $f(x) = 0$. But now the theorem follows, for if $f \in C(X)$, then

$$\lambda(f - \lambda(f) \cdot 1) = \lambda(f) - \lambda(f) \cdot \lambda(1) = 0.$$

By the above, we then have that $f(x) - \lambda(f) = 0$, so that $\lambda(f) = f(x)$. In other words, $\lambda = \mathrm{ev}_x$. ■

39. Mon, Dec. 2

Recently, we consider topological manifolds, which are a nice collection of spaces. Next semester, we will often work with another nice collection of spaces that can be built inductively. These are cell complexes, or CW complexes.

A typical example is a sphere. In dimension 1, we have S^1, which we can represent as the quotient of $I = [0, 1]$ by endpoint identification. Another way to say this is that we start with a point, and we "attach" an interval to that point by gluing both ends to the given point.

For S^2, there are several possibilities. One is to start with a point and glue a disk to the point (glueing the boundary to the point). An alternative is to start with a point, then attach an interval to get a circle. To this circle, we can attach a disk, but this just gives us a disk again, which we think of as a hemisphere. If we then attach a second disk (the other hemisphere), we get S^2.

But what do we really mean by "attach a disk"?

Let's start today by discussing the general "pushout" construction, which showed up last time in our construction of graphs.

Definition 39.1. Suppose that $f : A \longrightarrow X$ and $g : A \longrightarrow Y$ are continuous maps. The **pushout** (or glueing construction) of X and Y along A is defined as

$$X \cup_A Y := X \amalg Y / \sim, \qquad f(a) \sim g(a).$$

We have an inclusion $X \hookrightarrow X \amalg Y$. Composing this with the quotient map to $X \cup_A Y$ gives the map $\iota_X : X \longrightarrow X \cup_A Y$. We similarly have a map $\iota_Y : Y \longrightarrow X \cup_A Y$. Moreover, these maps make the diagram to the right commute. The point is that

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\iota_Y"] \\ X \arrow[r, "\iota_X"] & X \cup_A Y \end{tikzcd}$$

$$\iota_X(f(a)) = \overline{f(a)} = \overline{g(a)} = \iota_Y(g(a)).$$

For any n, we have the standard inclusion $S^{n-1} \hookrightarrow D^n$ as the boundary.

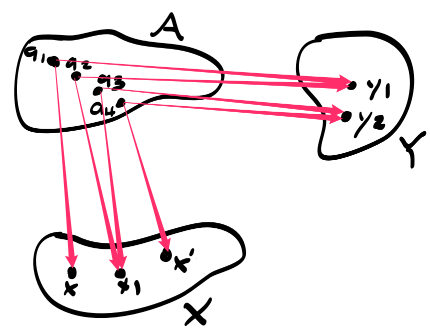
Definition 39.2. Given a space X and a continuous map $\alpha : S^{n-1} \longrightarrow X$, we write $X \cup_\alpha D^n$ for the pushout

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, "\alpha"'] & D^n \arrow[d] \\ X \arrow[r, "\iota_X"] & X \cup_\alpha D^n \end{tikzcd}$$

The image $\iota(\mathrm{Int}(D^n))$ is referred to as an n-cell and is sometimes denoted e^n. Thus the above space, which is described as obtained by attaching an n-cell to X, is also written $X \cup_\alpha \overline{e^n}$.

In general, this attaching process does not disturb the interiors of the cells, as follows from

Proposition 39.3. *If $g : A \hookrightarrow Y$ is injective, then $\iota_X : X \longrightarrow X \cup_A Y$ is also injective.*



Proof. Suppose that $\iota_X(x) = \iota_X(x')$. The relation imposed on $X \amalg Y$ only affects points in $f(A)$ and $g(A)$. We assume that $x, x' \in f(A)$ since otherwise we must have $x = x'$. In general, the situation we should expect is represented in the picture to the right. But since g is injective, this means that $a_1 = a_2$ and $a_3 = a_4$. This implies that $x = f(a_1) = f(a_2) = x_1$ and that $x_1 = f(a_3) = f(a_4) = x'$. Putting these together gives $x = x'$. ■

Example 39.4. If $A = \emptyset$, then $X \cup_A Y = X \amalg Y$.

Example 39.5. If $A = *$, then $X \cup_A Y = X \vee Y$.

The main point of this construction is the following property.

Proposition 39.6 (Universal property of the pushout)**.** *Suppose that $\varphi_1 : X \longrightarrow Z$ and $\varphi_2 : Y \longrightarrow Z$ are maps such that $\varphi_1 \circ f = \varphi_2 \circ g$. Then there is a unique map $\Phi : X \cup_A Y \longrightarrow Z$ which makes the diagram to the right commute.*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[d, "\iota_Y"] \arrow[ddr, bend left, "\varphi_2"] & \\ X \arrow[r, "\iota_X"] \arrow[drr, bend right, "\varphi_1"'] & X \cup_A Y \arrow[dr, dashed, "\Phi"] & \\ & & Z \end{tikzcd}$$

This generalizes the "pasting" lemma. Suppose that $U, V \subseteq X$ are open subsets. Then it is not difficult to show that the pushout $U \cup_{U \cap V} V$ is homeomorphic to X. The universal property for the pushout then says that specifying a continuous map out of X is the same as specifying a pair of continuous maps out of U and V which agree on their intersection $U \cap V$. This is precisely the statement of the pasting lemma!

By the way, Proposition 39.3 is not only true for injections.

Proposition 39.7. *(i) If $f : A \longrightarrow X$ is surjective, then so is $\iota_Y : Y \longrightarrow X \cup_A Y$.*

(ii) If $f : A \longrightarrow X$ is a homeomorphism, then so is $\iota_Y : Y \longrightarrow X \cup_A Y$.

Proof. We prove only (ii). We show that if f is a homeomorphism, then Y satisfies the same universal property as the pushout. Consider the test diagram to the right. We have no choice but to set $\Phi = \varphi_2$. Does this make the diagram commute? We need to check that $\Phi \circ g \circ f^{-1} = \varphi_1$. Well,

$$\Phi \circ g \circ f^{-1} = \varphi_2 \circ g \circ f^{-1} = \varphi_1 \circ f \circ f^{-1} = \varphi_1.$$

$$\begin{tikzcd}
A \arrow[rr, "g"] \arrow[d, "f"'] & & Y \arrow[d, equal] \arrow[ddr, bend left, "\varphi_2"] \\
X \arrow[r, "f^{-1}"] \arrow[drrr, bend right, "\varphi_1"'] & A \arrow[r, "g"] & Y \arrow[dr, dashed, "\Phi"] \\
& & & Z
\end{tikzcd}$$

■

40. Wed, Dec. 4

We use the idea of attaching cells (using a pushout) to inductively build up the idea of a cell complex or CW complex.

Definition 40.1. A **CW complex** is a space built in the following way

(1) Start with a discrete set X^0 (called the set of 0-cells, or the 0-skeleton)
(2) Given the $(n-1)$-skeleton X^{n-1}, the n-skeleton X^n is obtained by attaching n-cells to X^{n-1}.
(3) The space X is the union of the X^n, topologized using the "weak topology". This means that $U \subseteq X$ is open if and only if $U \cap X^n$ is open for all n.

A **cell complex** is defined similarly, except that we do not restrict the dimension of the cells attached at each stage.

The third condition is not needed if $X = X^n$ for some n (so that X has no cells in higher dimensions). On the other hand, the 'W' in the name CW complex refers to item 3 ("weak topology"). The 'C' in CW complex refers to the **C**losure finite property: the closure of any cell is contained in a finite union of cells. We will come back to this point later.

Example 40.2. (1) S^n. We have already discussed two CW structures on S^2. The first has X^0 a singleton and a single n-cell attached. The other had a single 0-cell and single 1-cell but two 2-cells attached. There is a third option, which is to start with two 0-cells, attach two 1-cells to get a circle, and then attach two 2-cells to get S^2.

The first and third CW structures generalize to any S^n. There is a minimal CW structure having a single 0-cell and single n-cell, and there is another CW structure have two cells in every dimension up to n.

(2) $\mathbb{RP}^n$. Let's start with $\mathbb{RP}^2$. Recall that one model for this space was as the quotient of D^2, where we imposed the relation $x \sim -x$ on the boundary. If we restrict our attention to the boundary S^1, then the resulting quotient is $\mathbb{RP}^1$, which is again a circle. The quotient map $q : S^1 \longrightarrow S^1$ is the map that winds twice around the circle. In complex coordinates, this would be $z \mapsto z^2$. The above says that we can represent $\mathbb{RP}^2$ as the pushout

$$\begin{tikzcd}
S^1 \arrow[r, "\iota"] \arrow[d, two heads, "q"'] & D^2 \arrow[d, two heads] \\
S^1 \arrow[r] & \mathbb{RP}^2
\end{tikzcd}$$

If we build the 1-skeleton S^1 using a single 0-cell and a single 1-cell, then $\mathbb{RP}^2$ has a single cell in dimensions ≤ 2.

More generally, we can define $\mathbb{RP}^n$ as a quotient of D^n by the relation $x \sim -x$ on the boundary S^{n-1}. This quotient space of the boundary was our original definition of $\mathbb{RP}^{n-1}$.

It follows that we can describe $\mathbb{RP}^n$ as the pushout

$$\begin{tikzcd}
S^{n-1} \arrow[r, "\iota"] \arrow[d, "q"'] & D^n \arrow[d] \\
\mathbb{RP}^{n-1} \arrow[r] & \mathbb{RP}^n
\end{tikzcd}$$

Thus $\mathbb{RP}^n$ can be built as a CW complex with a single cell in each dimension $\leq n$.

41. Fri, Dec. 6

(3) $\mathbb{CP}^n$. Recall that $\mathbb{CP}^1 \cong S^2$. We can think of this as having a single 0-cell and a single 2-cell. We defined $\mathbb{CP}^2$ as the quotient of S^3 by an action of S^1 (thought of as $U(1)$). Let $\eta : S^3 \longrightarrow \mathbb{CP}^1$ be the quotient map. What space do we get by attaching a 4-cell to $\mathbb{CP}^1$ by the map η? Well, the map η is a quotient, so the pushout $\mathbb{CP}^1 \cup_\eta D^4$ is a quotient of D^4 by the S^1-action on the boundary.

Now include D^4 into $S^5 \subseteq \mathbb{C}^3$ via the map

$$\varphi(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4, \sqrt{1 - \sum x_i^2}, 0).$$

(This would be a hemi-equator.) We have the diagonal $U(1)$ action on S^5. But since any nonzero complex number can be rotated onto the positive x-axis, the image of φ meets every S^1-orbit in S^5, and this inclusion induces a homeomorphism on orbit spaces

$$D^4/U(1) \quad \cong S^5/U(1) = \mathbb{CP}^2.$$

We have shown that $\mathbb{CP}^2$ has a cell structure with a single 0-cell, 2-cell, and 4-cell.

This story of course generalizes to show that any $\mathbb{CP}^n$ can be built as a CW complex having a cell in each even dimension.

(4) (Torus) In general, a product of two CW complexes becomes a CW complex. We will describe this in the case $S^1 \times S^1$, where S^1 is built using a single 0-cell and single 1-cell.

Start with a single 0-cell, and attach two 1-cells. This gives $S^1 \vee S^1$. Now attach a single 2-cell to the 1-skeleton via the attaching map ψ defined as follows. Let us refer to the two circles in $S^1 \vee S^1$ as ℓ and r. We then specify $\psi : S^1 \longrightarrow S^1 \vee S^1$ by $\ell r \ell^{-1} r^{-1}$. What we mean is to trace out ℓ on the first quarter of the domain, to trace out r on the second quarter, to run ℓ in reverse on the third quarter, and finally to run r in reverse on the final quarter.

We claim that the resulting CW complex X is the torus. Since the attaching map $\psi : S^1 \longrightarrow S^1 \vee S^1$ is surjective, so is $\iota_{D^2} : D^2 \longrightarrow X$. Even better, it is a quotient map. On the other hand, we also have a quotient map $I^2 \longrightarrow T^2$, and using the homeomorphism $I^2 \cong D^2$ from before, we can see that the quotient relation in the two cases agrees. We say that this homeomorphism $T^2 \cong X$ puts a cell structure on the torus. There is a single 0-cell (a vertex), two 1-cells (the two circles in $S^1 \vee S^1$), and a single 2-cell.

Let's talk about some of the (nice!) topological properties of CW complexes.

Lemma 41.1. *Let $I = \{e_i^{n_i}\}$ be the set of all cells in X. Then X is a quotient of $\coprod_i D^{n_i}$. In particular, $A \subseteq X$ is open (or closed) if and only if, for each cell i and corresponding characteristic map $\Phi_i : D^{n_i} \longrightarrow X$, the preimage $\Phi_i^{-1}(A)$ is open (or closed) in D^{n_i}.*

By the way, for 0-cells, we have that $e^0 = D^0$ is a point. In this case, $D^0 = \mathbb{R}^0$, so it is its own interior.

Proof. The forward implication is clear by continuity of the φ_i. For the other direction, suppose that each $\Phi_i^{-1}(A)$ is open. Then $A \cap X^0$ is open in X^0, since X^0 is just the disjoint union of its cells. Now assume by induction that $A \cap X^{n-1}$ is open in X^{n-1}. But, by the construction of the pushout, the n-skeleton X^n is a quotient of $X^{n-1} \amalg \coprod D^n$. Since $A \cap X^n$ pulls back to an open set in each piece of this coproduct, it must be open in $A \cap X^n$ by the definition of the quotient topology. Now, since $A \cap X^n$ is open in X^n for all n, A is open in X by property W. ■

Theorem 41.2. *Any CW complex X is normal.*

Proof. First, X is T_1 by the Lemma since any point obviously pulls back to a closed subset of every D_i^n. Let A and B be disjoint closed sets in X. We will show that X is normal by building a Urysohn function $f : X \longrightarrow [0, 1]$ with $f(A) \equiv 0$ and $f(B) \equiv 1$. Because X satisfies property W, a function f defined on X is continuous if and only if its restriction to each X^n is continuous. We thus build the function f by building its restrictions f^n to X^n.

On X^0, we define

$$f^0(x) = \begin{cases} 0 & x \in A \cap X^0 \\ 1 & x \in B \cap X^0 \\ 1/2 & \text{else.} \end{cases}$$

Since X^0 is discrete, this is automatically continuous.

Next time, we will give the induction step of building f^n from f^{n-1}.