Now assume by induction that we have $f^{n-1} : X^{n-1} \longrightarrow [0,1]$ continuous with $f^{n-1}(A \cap X^{n-1}) \equiv 0$ and $f^{n-1}(B \cap X^{n-1}) \equiv 1$. Since we have a pushout diagram

$$\begin{tikzcd} \coprod S^{n-1} \arrow[r] \arrow[d] & \coprod D^n \arrow[d] \\ X^{n-1} \arrow[r] & X^n, \end{tikzcd}$$

by the universal property of the pushout, to define f^n on X^n, we need only specify a compatible pair of functions on X^{n-1} and on the disjoint union. On X^{n-1}, we take f^{n-1}. To define a map out of $\coprod D^n$, it is enough to define a map on each D^n.

For each n-cell e^i, define $W_i \subseteq D^n$ closed by $W_i = \partial D^n \cup \Phi_i^{-1}(A \cap X^n) \cup \Phi_i^{-1}(B \cap X^n)$. Define $g : W_i \longrightarrow [0,1]$ by

$$g(x) = \begin{cases} f^{n-1}(\varphi(x)) & x \in \partial D^n \\ 0 & x \in \Phi_i^{-1}(A \cap X^n) \\ 1 & x \in \Phi_i^{-1}(B \cap X^n). \end{cases}$$

We know that D^n is compact Hausdorff (or metric) and thus normal. Thus, by the Tietze extneion theorem (31.1) there is a Urysohn function for the disjoint closed sets $\Phi_i^{-1}(A\cap X^n)$ and $\Phi_i^{-1}(B\cap X^n)$ whose restriction to ∂D^n agrees with $f^{n-1} \circ \varphi_i$. Putting all of this together gives a Urysohn function on X^n for the $A \cap X^n$ and $B \cap X^n$. By induction, we are done. ∎

Proposition 42.1. *Any CW complex X is locally path-connected.*

Proof. Let $x \in X$ and let U be any open neighborhood of x. We want to find a path-connected neighborhood V of x in U. Recall that a subset $V \subseteq X$ is open if and only if $V \cap X^n$ is open for all n. We will define V by specifying open subsets $V^n \subseteq X^n$ with $V^{n+1} \cap X^n = V^n$ and then setting $V = \cup V^n$.

Suppose that x is contained in the cell e_i^n. We set $V^k = \emptyset$ for $k < n$. We specify V_n by defining $\Phi_j^{-1}(V^n)$ for each n-cell e_j^n. If $j \neq i$, we set $\Phi_j^{-1}(V_n) = \emptyset$. We define $\Phi_i^{-1}(V_n)$ to be an open n-disc around $\Phi_i^{-1}(x)$ whose closure is contained in $\Phi_i^{-1}(U)$. Now suppose we have defined V^k for some $k \geq n$. Again, we define V^{k+1} by defining each $\Phi_i^{-1}(V^{k+1})$. By assumption, $\overline{\Phi_i^{-1}(V^k)} \subseteq \partial D^{k+1} \subseteq \Phi_i^{-1}(U)$. By the Tube lemma, there is an $\epsilon > 0$ such that (using radial coordinates) $\Phi_i^{-1}(V^k) \times (1-\epsilon, 1] \subset U$. We define

$$\Phi_i^{-1}(V^{k+1}) = \Phi_i^{-1}(V^k) \times [1, 1-\epsilon/2),$$

which is path-connected by induction. This also guarantees that $\overline{V^{k+1}} \subset U \cap X^{k+1}$, allowing the induction to proceed. ∎

Proposition 42.2 (Hatcher, A.1)**.** *Any compact subset K of a CW complex X meets finitely many cells.*

Proof. For each cell e_i meeting K, pick a point $k_i \in K \cap e_i$. Let $S = \{k_i\}$. We use proprety W to show that S is closed in X. It is clear that $S \cap X^0$ is closed in X^0 since X^0 is discrete. Assume that $S \cap X^{n-1}$ is closed in X^{n-1}. Now in X^n, the set $S \cap X^n$ is the union of the closed subset $S \cap X^{n-1}$ and the points k_i that lie in open n-cells. By Lemma 41.1, this set of k_i is closed as well.

The argument above in fact shows that any subset of S is closed, so that S is discrete. But S is closed in K, so S is compact. Since S is both discrete and compact, it must be finite. ∎

Corollary 42.3. *Any CW complex has the closure-finite property, meaning that the closure of any cell meets finitely many cells.*

Proof. The closure of e_i is $\Phi_i(D_i^{n_i})$, which is compact. The result follows from the proposition. ■

43. Wed. Dec. 11

Proposition 43.1. *A CW complex X is locally compact if and only if every point has a neighbohood that meets finitely many cells.*

Theorem 43.2 (Lee, 5.25)**.** *Every 1-manifold admits a (nice) CW decomposition.*

Theorem 43.3 (Lee, 5.36, 5.37)**.** *Every n-manifold admits a (nice) CW decomposition for $n = 2, 3$.*

According to p. 529 of Allen Hatcher's Algebraic Topology book, it is an open question whether or not every 4-manifold admits a CW decomposition. But n-manifolds for $n \geq 5$ do always admit a CW decomposition.

Theorem 43.4. *Every nonempty, connected 1-manifold M is homeomorphic to S^1 if it is compact and to $\mathbb{R}$ if it is noncompact.*

For this theorem, it will be convenient to work with nice CW structures.

Definition 43.5. If X is a space with a CW structure, we say that an n-cell e_i^n is **regular** if the characteristic map $\Phi_i : D^n \longrightarrow \overline{e_i} \subset X$ is a homeomorphism onto its image. We say that a CW complex is regular if every cell is regular.

Proof. The first step is to show that every 1-manifold has a regular CW decomposition. The main idea is to cover M by a countable collection $\{U_n\}$ of regular charts (each closure U_n in M should be homeomorphic to $[0, 1]$). Then, using induction, it is possible to a regular CW structure on $\mathcal{U}_n = \bigcup_{k=1}^n U_k$ in such a way that $\mathcal{U}_n \subseteq \mathcal{U}_{n+1}$ is the inclusion of a subcomplex. (See Lee 5.25 for more details.) Clearly, each 1-cell bounds two 0-cells, since the 1-cell is assume to be regular. Somewhat less clear is the fact that each 0-cell is in the boundary of two 1-cells (see Lee 5.26).

We enumerate the 0-cells (aka vertices) and 1-cells (aka edges) in the following way. First, pick some 0-cell, and call it v_0. Pick an edge ending at v_0, and call this e_0. The other endpoint of e_0 we call v_1. The other edge ending at v_1 is called e_1. We can continue in this way to get $v_2, v_3, \dots$ and $e_2, e_3, \dots$. Now there is also another edge ending at v_0, which should be called e_{-1}. Let v_{-1} be the other endpoint. We can continue to get $v_{-2}, v_{-3}, \dots$ and $e_{-2}, e_{-3}, \dots$.

There are two cases to consider:

Case 1: The vertices v_i, $i \in \mathbb{Z}$ are all distinct. Then for each $n \in \mathbb{Z}$, we have an embedding $[n, n+1] \cong [-1, 1] \xrightarrow{\Phi_n} \overline{e_n^1}$. These glue together to give a continuous map $f : \mathbb{R} \longrightarrow X$. Our assumption means that f is injective when restricted to $\mathbb{Z}$. We can then see it is globally injective since its restriction to any $(n, n+1)$ is a characteristic map for a cell (thus injective) and all cells are disjoint. Since M is connected and thus path-connected (M is locally Euclidean) any path from v_0 to any other point is contained in a finite union of the U_i's. It follows that $f : \mathbb{R} \longrightarrow M$ is surjective.

Finally, we show that f is open. Any open subset of $(n, n+1)$ is taken by f to an open subset of M, since the top-dimension cells are always open in a CW complex. It remains to show that f takes intervals of the form $(n - \epsilon, n + \epsilon)$ to open subsets of M. By taking ϵ small enough, we can ensure that this image is contained in (the closure of) two 1-cells. We can then see that this subset of M is open by pulling back along the characteristic maps (pulling back along these two characteristic maps will give half-open intervals in D^1).

Case 2: For some $n \in \mathbb{Z}$ and $k > 0$, we have $v_n = v_{n+k}$. We may then pick n and k so that k is minimal. Then the vertices $v_n, \ldots, v_{n+k-1}$ are distinct, as are the edges $e_n, \ldots, e_{n+k-1}$. This implies that the restriction of f to $[n, n+k)$ is injective. If we consider the restriction only to the closed interval $[n, n+k]$, then we get a closed map, since the domain is compact and the target is Hausdorff. We claim also that $f([n, n+k])$ is open in M. Indeed, if we pick any $x \in [n, n+k]$ which lies in an interval $(i, i+1)$, then the open 1-cell e_i is a neighborhood of $f(x)$ that is contained in the image of f. If we consider any interior integer $n < i < n+k$, then $e_{i-1} \cup \{v_i\} \cup e_i$ is an open neighborhood in the image of f. Finally, $e_n \cup \{n\} e_{n+k-1}$ is a neighborhood of $f(n) = f(n+k)$ in M.

Since the image $f([n, n+k])$ is both closed and open in M and M is connected, we conclude that $f([n, n+k]) = M$. Since $f(n) = f(n+k)$, we get an induced map

$$\overline{f} : [n, n+k]/\sim \;\cong S^1 \longrightarrow M$$

which is a bijection. Since S^1 is compact and M is Hausdorff, this is a homeomorphism. ■

We previously also briefly mentioned the idea of a "manifold with boundary". There is a similar result:

Theorem 44.1. *Every nonempty, connected* 1*-manifold with boundary is homeomorphic to* $[0, 1]$ *if it is compact and to* $[0, 1)$ *if it is noncompact.*

Next semester, we will similarly classify all compact 2-manifolds (the list of answers will be a little longer).

A closely related idea to CW complex is the notion of **simplicial complex**. A simplicial complex is built out of "simplices". By definition, an n**-simplex** is the convex hull of $n+1$ "affinely independent" points in $\mathbb{R}^k$, for $k \geq n+1$. This means that after translating this set so that one point moves to the origin, the resulting collection of points is linearly independent.

There is a standard n-simplex $\Delta^n \subseteq \mathbb{R}^{n+1}$ defined by

$$\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1, t_i \geq 0\}.$$

In general, if σ is an n-simplex generated by $\{t_0, \ldots t_n\}$, then the convex hull of any subset is called a **face** of the simplex. A (Euclidean) simplicial complex is then a subset of $\mathbb{R}^k$ that is a union of simplices such that any two overlapping simplices meet in a face of each. We also usually require the collection of simplices to be locally finite.

Since an n-simplex is homeomorphic to D^n, it can be seen that a simplicial complex is a regular CW complex. A decomposition of a manifold as a simplicial complex is known as a **triangulation** of the manifold. Just as one can ask about CW structures on manifolds, one can also ask about triangulations for manifolds.

Theorem 44.2.
1. *Every* 1*-manifold is triangulable (indeed, we know the complete list of connected* 1*-manifolds).*
2. *Tibor Radó proved in 1925 that every* 2*-manifold is triangulable.*
3. *Edwin Moise proved in the 1950s that every* 3*-manifold is triangulable.*
4. *Michael Freedman discovered the* 4*-dimensional* E_8*-manifold in 1982, which is not triangulable.*
5. *Ciprian Manolescu showed in March 2013 that manifolds in dimension* ≥ 5 *are not triangulable.*