3. Wed, Sept. 4

Recall that we were in the midst of proving the following result

Proposition 2.8. *Let $f : X \longrightarrow Y$ be a function between metric spaces. The following are equivalent:*

(1) *f is continuous*
(6) *For every convergent sequence $(x_n) \to x$ in X, the sequence $(f(x_n))$ converges to $f(x)$ in Y.*

Proof. It remains to show that (6) implies continuity. Do this on your homework! ∎

What constructions can we make with metric spaces?

Products: Let's start with a product. That is, if (X, d_X) and (Y, D_Y) are metric spaces, is there a good notion of the product metric space? We would want to have "projection" maps to each of X and Y, and we would want it to be true that to define a continuous map from some metric space Z to the product, it is enough to specify continuous maps to each of X and Y. By thinking about the case in which Z has a discrete metric, one can see that the underlying set of the product metric space would need to be the cartesian product $X \times Y$. The only question is whether or not there is a sensible metric to define.

Recall that we discussed three metrics on $\mathbb{R}^2$: the standard one, the max metric, and the taxicab metric. There, we used that $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ as an underlying set, and we combined the metrics on each copy of $\mathbb{R}$ to get a metric on $\mathbb{R}^2$. We can use the same idea here to get three different metrics on $X \times Y$, and these will all produce a metric space satisfying the right property to be a product.

Function spaces: Another important construction is that of a space of functions. That is, if X and Y are metric spaces, one can consider the set of all continuous functions $f : X \longrightarrow Y$. Is there a good way to think of this as a metric space? For example, as a set $\mathbb{R}^2$ is the same as the collection of functions $\{1, 2\} \longrightarrow \mathbb{R}$. More generally, we could consider functions $\{1, \dots, n\} \longrightarrow Y$ or even $\mathbb{N} \longrightarrow Y$ (i.e. sequences).

Of the metrics we discussed on $\mathbb{R}^2$, the max metric generalizes most easily to give a metric on $Y^\infty = Y^{\mathbb{N}}$. We provisionally define the **sup metric** on the set of sequences in Y by

$$d_{\sup}((y_n), (z_n)) = \sup_n \{d_Y(y_n, z_n)\}.$$

Without any further restrictions, there is no reason that this supremum should always exist. If Y is a bounded metric space, or if we only consider bounded sequences, then we are OK. Another option is to arbitrarily truncate the metric.

Lemma 3.1. *Let (Y, d) be a metric space. Define the resulting bounded metric $\overline{d}$ on Y by*

$$\overline{d}(y, z) = \max\{d(y, z), 1\}.$$

This is a metric, and the open sets determined by $\overline{d}$ are precisely the open sets determined by d.

We now redefine the sup metric on Y^∞ to be

$$d_{\sup}((y_n), (z_n)) = \sup_n \{\overline{d_Y}(y_n, z_n)\}.$$

Now the supremum always exists, so that we get a well-defined metric. The same definition works to give a metric on the set of continuous functions $X \longrightarrow Y$. We define the sup metric on the set $\mathcal{C}(X, Y)$ of continuous functions to be

$$d_{\sup}(f, g) = \sup_{x \in X} \{\overline{d_Y}(f(x), g(x))\}.$$

This is also called the **uniform metric**, for the following reason.

Proposition 3.2. *Let (f_n) be a sequence in $\mathcal{C}(X,Y)$. Then $(f_n) \to f$ in the uniform metric on $\mathcal{C}(X,Y)$ if and only if $(f_n) \to f$ uniformly.*

Given a function $f \in \mathcal{C}(X,Y)$ and a point $x \in X$, one can evaluate the function to get $f(x) \in Y$. In other words, we have an evaluation function

$$\text{eval} : \mathcal{C}(X,Y) \times X \longrightarrow Y.$$

Proposition 3.3. *Consider $\mathcal{C}(X,Y) \times X$ as a metric space using the max metric. Then* eval *is continuous.*

Proof. We know that to determine if a function between metric spaces is continuous, it it suffices to check that it takes convergent sequences to convergent sequences. Suppose that $(f_n, x_n) \to (f, x)$. We wish to show that

$$\text{eval}(f_n, x_n) = f_n(x_n) \to \text{eval}(f, x) = f(x).$$

Let $\varepsilon > 0$. Since $(f_n, x_n) \to (f, x)$, it follows that $f_n \to f$ and $x_n \to x$ (since the projections are continuous. Then there exists N_1 such that if $n > N_1$ then $d_{\text{sup}}(f_n, f) < \varepsilon/2$. By the definition of the sup metric, this implies that $d_Y(f_n(x_n), f(x_n)) < \varepsilon/2$. But now f is continuous, so there exists N_2 such that if $n > N_2$ then $d_Y(f(x_n), f(x)) < \varepsilon/2$. Putting these together and using the triangle inequality, if $n > N_3 = \max\{N_1, N_2\}$ then $d_Y(f_n(x_n), f(x)) < \varepsilon$. ∎

Proposition 3.4. *Suppose $\varphi : X \times Y \longrightarrow Z$ is continuous. For each $x \in X$, define $\hat{\varphi}(x) : Y \longrightarrow Z$ by $\hat{\varphi}(x)(y) = \varphi(x, y)$. The function $\hat{\varphi}(x)$ is continuous.*

Proof. This could certainly be done directly, using convergence of sequences to test for continuity. Here is another way to do it, using the universal property of products.

Note that $\hat{\varphi}(x)$ can be written as the composition $Y \xrightarrow{i_x} X \times Y \xrightarrow{\varphi} Z$. By assumption, φ is continuous, so it suffices to know that $i_x : Y \to X \times Y$ is continuous. But recall that continuous maps into a product correspond precisely to a pair of continuous maps into each factor. The pair of maps here is the constant map $Y \longrightarrow X$ at x and the identity map $Y \longrightarrow Y$. Both are clearly continuous, so it follows that i_x is also continuous. ∎

We are headed to the universal property of the mapping space. Keeping the notation from above, given a continuous function

$$\varphi : X \times Y \longrightarrow Z,$$

we get a function

$$\hat{\varphi} : X \longrightarrow \mathcal{C}(Y, Z).$$

Conversely, given the function $\hat{\varphi}$, we define φ by

$$\varphi(x, y) = \hat{\varphi}(x)(y).$$

Proposition 3.5. *The function φ above is continuous if and only if $\hat{\varphi}$ is continuous.*

Proof. On homework 2. ∎

4. Fri, Sept. 6

Quotients Another important construction that we will discuss when we move on to topological spaces is that of a quotient, or identification space. A standard example is the identification, on the unit interval $[0, 1]$, of the two endpoints. Glueing these together gives a circle S^1, and the surjective continuous map

$$e^{2\pi i x} : [0, 1] \longrightarrow S^1$$

is called the quotient map. Here the desired universal property is that if $f : [0,1] \longrightarrow Y$ is a continuous map to another metric space such that $f(0) = f(1)$, then the map f should "factor" through the quotient. Quotients become quite complicated to express in the world of metric spaces.

Now that we have spent some time with metric spaces, let's turn to the more general world of topological spaces.

Definition 4.1. A **topological space** is a set X with a collection of subsets $\mathcal{T}$ of X such that

(1) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$
(2) If $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$
(3) If $U_i \in \mathcal{T}$ for all i in some index set I, then $\bigcup U_{i \in I} U_i \in \mathcal{T}$.

The collection $\mathcal{T}$ is called the **topology** on X, and the elements of $\mathcal{T}$ are referred to as the "open sets" in the topology.

Example 4.2. (1) (Metric topology) Any metric space is a topological space, where $\mathcal{T}$ is the collection of metric open sets
(2) (Discrete topology) In the discrete topology, *every* subset is open. We already saw the discrete metric on any set, so in fact this is an example of a metric topology as well.
(3) (Trivial topology) In the trivial topology, $\mathcal{T} = \{\emptyset, X\}$. That is, $\emptyset$ and X are the only empty sets. This topology does not come from a metric (unless X has fewer than two points).
(4) It is simple to write down various topologies on a finite set. For example, on the set
$$X = \{1, 2\},$$
there are 4 possible topologies. In addition to the trivial and discrete topologies, there is also
$$\mathcal{T}_1 = \{\emptyset, \{1\}, X\}$$
and
$$\mathcal{T}_2 = \{\emptyset, \{2\}, X\}.$$
(5) There are many possible topologies on $X = \{1, 2, 3\}$. But not every collection of subsets will give a topology. For instance,
$$\{\emptyset, \{1\}, \{1, 3\}, X\}$$
would not be a topology, since it is not closed under intersection.

When working with metric spaces, we saw that the topology was determined by the open balls. Namely, an open set was precisely a subset that could be written as a union of balls. In many topologies, there is an analogue of these basic open sets.

Definition 4.3. A **basis** for a topology on X is a collection $\mathcal{B}$ of subsets such that

(1) (Covering property) Every point of x lies in at least one basis element
(2) (Intersection property) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there is a third basis element B_3 such that
$$x \in B_3 \subseteq B_1 \cap B_2.$$

A basis $\mathcal{B}$ defines a topology $\mathcal{T}_{\mathcal{B}}$ by declaring the open sets to be the unions of (arbitrarily many) basis elements.

Proposition 4.4. *Given a basis $\mathcal{B}$, the collection $\mathcal{T}_{\mathcal{B}}$ is a topology.*

Proof. It is clear that open sets are closed under unions. The emptyset is a union of no basis elements, so it is open. The set X is open by the covering property: the union of all basis elements is X. Finally, we check that the intersection of two open sets is open. Let U_1 and U_2 be open. Then

$$U_1 = \bigcup_{\alpha \in A} B_\alpha, \qquad U_2 = \bigcup_{\delta \in \Delta} B_\delta.$$

We want to show that $U_1 \cap U_2$ is open. Now

$$U_1 \cap U_2 = \left(\bigcup_{\alpha \in A} B_\alpha \right) \cap \left(\bigcup_{\delta \in \Delta} B_\delta \right) = \bigcup_{\alpha \in A, \delta \in \Delta} B_\alpha \cap B_\delta.$$

It remains to show that $B_\alpha \cap B_\delta$ is open. By the intersection property of a basis, for each $x \in B_\alpha \cap B_\delta$, there is some B_x with

$$x \in B_x \subseteq B_\alpha \cap B_\delta.$$

It follows that

$$B_\alpha \cap B_\delta = \bigcup_{x \in B\alpha \cap B_\delta} B_x,$$

so we are done. ■

Example 4.5. We have already seen that metric balls form a basis for the metric topology. In the case of the discrete metric, one can take the balls with radius 1/2, which are exactly the singleton sets.

Example 4.6. For a truly new example, we take as basis on $\mathbb{R}$, the half-open intervals $[a, b)$. The resulting topology is known as the **lower limit topology** on $\mathbb{R}$.

How is this related to the usual topology on $\mathbb{R}$? Well, any open interval (a, b) can be written as a union of half-open intervals. However, the $[a, b)$ are certainly not open in the usual topology. This says that $\mathcal{T}_{\text{standard}} \subseteq \mathcal{T}_{\ell\ell}$. The lower limit topology has more open sets than the usual topology. When one topology on a set has more open sets than another, we say it is **finer**. So the lower limit topology is *finer* than the usual topology on $\mathbb{R}$, and the usual topology is *coarser* than the lower limit topology.

On any set X, the discrete topology is the finest, whereas the trivial topology is the coarsest.

When a topology is generated by a basis, there is a convenient criterion for open sets.

Proposition 4.7. *(Local criterion for open sets) Let $\mathcal{T}_\mathcal{B}$ be a topology on X generated by a basis $\mathcal{B}$. Then a set $U \subseteq X$ is open if and only if, for each $x \in U$, there is a basis element $B_x \in \mathcal{B}$ with $x \in B_x \subseteq U$.*

Proof. ($\Rightarrow$) By definition of $\mathcal{T}_\mathcal{B}$, the set U is a union of basis elements, so any $x \in U$ must be contained in one of these.

($\Leftarrow$) We can write $U = \bigcup_{x \in U} B_x$. ■

This is a good time to introduce a convenient piece of terminology: given a point x of a space X, a **neighborhood** N of x in X is a subset of X containing some open set U with $x \in U \subseteq N$. Often, we will take our neighborhoods to themselves be open.