8. Mon, Sept. 16

The last few lectures, we have seen that closed sets are not as easily understood in general as they are in the case of metric spaces. Although we will not want to restrict ourselves to metric spaces, it will nevertheless be helpful to have some good characterizations of the ”reasonable” spaces. We mention here a few of these properties.

Definition 8.1. A space X is said to be Hausdorff (also called T_2) if, given any two points x and y in X, there are disjoint open sets U and V with $x \in U$ and $y \in V$.

This is a somewhat mild “separation property” that is held by many spaces in practice and that also has a number of nice consequences.

Proposition 8.2. *If X is Hausdorff, then points are closed in X.*

Proof. The neighborhood crtierion for the complement $X \setminus \{x\}$ is easy to verify. ■

The Hausdorff property forces sequences to behave well, in the following sense.

Proposition 8.3. *In a Hausdorff space, a sequence cannot converge simultaneously to more than one point.*

Proof. Suppose $x_n \to x$ and $x_n \to y$. Every neighborhood of x contains a tail of x_n, as does any neighborhood of y. It follows that no neighborhood of x is disjoint from any neighborhood of y. Since X is Hausdorff, this forces $x = y$. ■

Proposition 8.4. *Every metric space is Hausdorff.*

Proof. If $x \neq y$, let $d = d(x, y) > 0$. Then the balls of radius $d/2$ centered at x and y are the needed disjoint neighborhoods. ■

However, of the (many, many) topologies on a finite set, the only one that is Hausdorff is the discrete topology. Indeed, if points are closed, then every subset is closed, as it is a finite union of points.

Another property of metric spaces that we used recently was the existence of the balls of radius $1/n$.

Definition 8.5. A space X is **first-countable** if, for each $x \in X$, there is a countable collection $\{U_n\}$ of neighborhoods of x such that any other neighborhood contains at least on of the U_n.

This was the key property used in proving that, in a metric space, an acuumulation point of $A \subseteq X$ is the limit of an A-sequence.

Example 8.6. The space $X = \mathbb{R}_{\text{cocountable}}$ is not first countable. To see this, let $x \in X$ and suppose that $\{U_n\}$ is a collection of neighborhoods of x. By definition, each U_n is open and misses only countably many real numbers. Write $C_n = \mathbb{R} \setminus U_n$. Then $C = \bigcup_n C_n$ is also countable, and it follows that $U = X \setminus C$ is a neighborhood of x. But U does not contain any U_n because if $U_n \subseteq U$, this would mean that $C_n \supseteq C$. Instead, we see that $C_n \subseteq C$, so that $U \subseteq U_n$ for all n. The above argument is not quite careful enough, since all of the above inclusions could be equalities. To fix it, simply note that the countable set $\bigcup_n C_n$ cannot be all of $\mathbb{R}$, since it is countable. Let C' be the union C, but with one extra element of $\mathbb{R}$ added in. Then C' is still countable, and each C_n is strictly contained in C.

We will return to first-countable (and second-countable) spaces later in the course.

In Calculus, you saw functions defined piecewise, and one-sided limits were typically employed to establish continuity. There is an analogue of this type of construction for spaces.

Lemma 8.7 (Glueing/Pasting Lemma)**.** *Let $X = A \cup B$, and suppose given a function $f : X \longrightarrow Y$. Assume that the restrictions $f_{|A}$ and $f_{|B}$ are both continuous. Then f is continuous, provided that either (1) both A and B are open in X or (2) both A and B are closed in X.*

Proof. We give the proof assuming they are both open. Let $V \subseteq Y$ be open. We wish to show that $f^{-1}(V) \subseteq X$ is open. Let's restrict to A. We have $f^{-1}(V) \cap A = f_{|A}^{-1}(V)$. Since $f_{|A}$ is continuous, it follows that $f_{|A}^{-1}(V)$ is open (in A). Since A is open in X, it follows that $f_{|A}^{-1}(V)$ is also open in X. The same argument shows that $f^1(V) \cap B$ is open in X. It follows that their unoin, which is $f^{-1}(V)$, is open in X. ■

Example 8.8. For example, we can use this to paste together the continuous absolute value function $f(x) = |x|$, as a function $\mathbb{R} \longrightarrow \mathbb{R}$. We get this by pasting the continuous functions $\iota : [0, \infty) \longrightarrow \mathbb{R}$, $x \mapsto x$, and $(-\infty, 0] \cong [0, \infty) \longrightarrow \mathbb{R}$, $x \mapsto -x$.

Example 8.9. Looking at an example of a discontinuous function for example

$$f(x) = \begin{cases} 1 & x \neq 1 \\ 2 & x = 1, \end{cases}$$

we can get this by pasting together two constant functions, but the domains are $\mathbb{R} \setminus \{1\}$ and $\{1\}$, one of which is open but not closed, and the other of which is closed but not open.

Finally, we start to look at the idea of sameness. Two sets are thought of as the same if there is a bijection between them. A bijection is simply an invertible function. More generally, we have the following idea.

Definition 8.10. A "morphism" $f : X \longrightarrow Y$ is said to be an **isomorphism** if there is a $g : Y \longrightarrow X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Again, an isomorphism between sets is simply a bijection. In topology, this is called a **homeomorphism**. In other words, a homeomorphism is a continuous function with a continuous inverse.

9. Wed, Sept. 18

Last time, we discussed the glueing lemma. Let's look at one more example of this in use.

Example 9.1. Let $X = [0, 1] \cup [2, 3]$, given the subspace topology from $\mathbb{R}$. Note that in this case each of the subsets $A = [0, 1]$ and $B = [2, 3]$ is **both** open an closed, so we can specify a continuous function on X by giving a pair of continuous functions, one on A and the other on B.

We introduced the concept of **homeomorphism** at the end of class last time, which is simply a continuous map with a continuous inverse. Since such a map is invertible, clearly it must be one-to-one and onto, but it is **not** true that every continuous bijection is a homeomorphism. Before we look at some examples, let's look at some non-examples.

Example 9.2. (1) Any time a set is equipped with two topologies, one of which is a refinement of the other, the identity map is a continuous bijection (in one direction) that is not a homeomorphism. For instance, we have the following such examples

$$\mathrm{id} : \mathbb{R} \longrightarrow \mathbb{R}_{\text{cofinite}}, \qquad \mathrm{id} : \mathbb{R}_{\text{cocountable}} \longrightarrow \mathbb{R}_{\text{cofinite}} \qquad \mathrm{id} : \mathbb{R}_{\text{discrete}} \longrightarrow \mathbb{R}$$

(2) Consider the exponential map $\exp : [0,1) \longrightarrow S^1$ given by $\exp(x) = e^{2\pi i x}$. This is a continuous bijection, but it is not a homeomorphism. Since homeomorphisms have continuous inverses, they must take open sets to open sets and closed sets to closed sets. But we see that exp does not take the open set $U = [0, 1/2)$ to an open set in S^1. The point $\exp(0) = (1,0)$ has no neighborhood that is contained in $\exp(U)$.

In the last example, we noted that homeomorphisms must take open sets to open sets. Such a map is called an **open map**. Similarly, a **closed map** takes closed sets to closed sets.

Proposition 9.3. *Let $f : X \longrightarrow Y$ be a continuous bijection. The following are equivalent:*

(1) *f is a homeomorphism*
(2) *f is an open map*
(3) *f is a closed map*

If we drop the assumption that f is bijective, it is no longer true that being an open map is equivalent to being a closed map. For example, the inclusion $(0,1) \longrightarrow \mathbb{R}$ is open but not closed, and the inclusion $[0,1] \longrightarrow \mathbb{R}$ is closed but not open.

Example 9.4. (1) Consider $\arctan : (0, \frac{\pi}{2}) \longrightarrow (0,\infty)$. This is a continuous bijection with continuous inverse (given by tangent)

(2) Consider $\ln : (0,\infty) \longrightarrow \mathbb{R}$. This is a continuous bijection with inverse e^x. Composing homeomorphisms produces homeomorphisms, and we therefore get a homeomorphism

$$(0,1) \xrightarrow{\cong} (0, \frac{\pi}{2}) \xrightarrow{\cong} (0,\infty) \xrightarrow{\cong} \mathbb{R}.$$

(3) We similarly get a homeomorphism $\arctan : [0, \frac{\pi}{2}) : \xrightarrow{\cong} [0,\infty)$. It follows that we have

$$[0,1) \cong [0,\infty) \qquad \text{and} \qquad (0,1] \cong [0,\infty).$$

(4) One can similarly get $B_r^n(x) \cong \mathbb{R}^n$ for any n, r, and x.

The above example shows that there really are only three intervals, up to homeomorphism: the open interval, the half-open interval, and the closed interval.

We say that two spaces are **homeomorphic** if there is a homeomorphism between them (and write $X \cong Y$ as aboce). This is the notion of "sameness' for spaces. One of the major overarching questions for this course will be: how can we tell when two spaces are the same or are actually different?

A standard way to show that two spaces are not homeomorphic is to find a property that one has and the other does not. For instance every metric space is Hausdorff, so non non-Hausdorff space is the "same" as a metric space. But what property distinguishes the 3 interval types above? As we learn about more and more properties of spaces, this question will become easier to answer.

We turn now to the construction phase. We considered the product of metric spaces: let's define the product for spaces. We already know what property it should satisfy: we want it to be true that mapping continuously from some space Z into the product $X \times Y$ should be the same as mapping separately to X and to Y. Another way to describe this is that we want $X \times Y$ to be the "universal" example of a space with a pairs of maps to X and Y.

Well, if the projection $p_X : X \times Y \longrightarrow X$ is to be continuous, we need $p_X^{-1}(U) = U \times Y$ to be open whenever $U \subseteq X$ is open. Similarly, we need $X \times V$ to be open if $V \subseteq Y$ is open. We are forced to include these open sets, but we don't want to throw in anything extra that we don't need. In other words, we want the product topology on $X \times Y$ to be the coarsest topology containing the sets $U \times Y$ and $X \times V$.

Note that if we consider the collecion

$$\mathcal{B} = \{U \times Y\} \cup \{X \times V\},$$

this cannot be a basis because it fails the intersection property. A typical intersection is

$$(U \times Y) \cap (X \times V) = U \times V,$$

and if we consider all sets of this form, we do get a basis.

Definition 9.5. Given spaces X and Y, the product topology on $X \times Y$ has basis given by sets of the form $U \times V$, where U and V are open in X and Y, respectively.

This satisfies the universal property of a product. We have engineered the definition to make this so, but we will check this next time anyway.

We pointed out above that if we considered the collection

$$\mathcal{B} = \{U \times Y\} \cup \{X \times V\},$$

we would not have a basis, as the intersection property failed. We remedied this by considering instead intersections of elements of $\mathcal{B}$. This is a useful idea that shows up often.

Given a set X, a collection $\mathcal{C}$ of subsets of X is called a **subbasis** for a topology on X if the collection covers X.

10. Fri, Sept. 20

We can then get a basis from the subbasis by considering finite intersections of subbasis elements.

Example 10.1. The collection of rays (a, ∞) and $(-\infty, b)$ give a subbasis for the standard topology on $\mathbb{R}$.

We introduced the product topology last time and mentioned the universal property, but let's spend a little bit of time with it today to really nail down the concept.

Theorem-Definition 10.2. *Let X and Y be spaces. Then $X \times Y$, together with the projection maps*

$$p_X : X \times Y \longrightarrow X \quad \text{and} \quad p_Y : X \times Y \longrightarrow Y,$$

satisfies the following "universal property": given any space Z and maps $g : Z \longrightarrow X$ and $h : Z \longrightarrow Y$, there is a unique continuous map $f : Z \longrightarrow X \times Y$ such that

$$g = p_X \circ f, \qquad h = p_Y \circ f.$$

$$\begin{tikzcd}
& & X \\
Z \arrow[urr, bend left, "g"] \arrow[r, dashed, "\exists! f"] \arrow[drr, bend right, "h"'] & X \times Y \arrow[ur, "p_X"'] \arrow[dr, "p_Y"] & \\
& & Y
\end{tikzcd}$$

Proof. The uniqueness is clear: if there exists such a continuous map f, then the conditions force this to be $f = (g, h)$. The only question is whether or not f is continuous. Consider a typical basis element $U \times V$ for the product topology on $X \times Y$. Then

$$\begin{aligned} f^{-1}(U \times V) = \{z \in Z \mid f(z) \in U \times V\} &= \{z \in Z \mid g(z) \in U \text{ and } h(z) \in V\} \\ &= g^{-1}(U) \cap h^{-1}(V), \end{aligned}$$

which is an intersection of open sets and therefore open. ∎

Ok, so we showed that $X \times Y$ satisfies this property, but why do we call this a "universal property"?

Proposition 10.3. *Suppose W is a space with continuous maps $q_X : W \longrightarrow X$ and $q_Y : W \longrightarrow Y$ also satisfying the property of the product. Then W is homeomorphic to $X \times Y$.*

Proof. The universal property for $X \times Y$ gives us a map $f : W \longrightarrow X \times Y$.

$$\begin{tikzcd}
 & & X \\
W \arrow[r, dashed, "\exists! f"] \arrow[urr, bend left, "q_X"] \arrow[drr, bend right, "q_Y"'] & X \times Y \arrow[ur, "p_X"'] \arrow[dr, "p_Y"] & \\
 & & Y
\end{tikzcd}$$

But W also has a universal property, so we get a map $\varphi : X \times Y \longrightarrow W$ as well.

$$\begin{tikzcd}
 & & X \\
X \times Y \arrow[r, dashed, "\exists! \varphi"] \arrow[urr, bend left, "p_X"] \arrow[drr, bend right, "p_Y"'] & W \arrow[ur, "q_X"'] \arrow[dr, "q_Y"] & \\
 & & Y
\end{tikzcd}$$

Now make Pacman eat Pacman!

$$\begin{tikzcd}
 & & & X \\
W \arrow[r, dotted, "f"] \arrow[urrr, bend left, "q_X"] \arrow[drrr, bend right, "q_Y"'] & X \times Y \arrow[r, dotted, "\varphi"] \arrow[urr, dotted, "p_X"] \arrow[drr, dotted, "p_Y"'] & W \arrow[ur, "q_X"'] \arrow[dr, "q_Y"] & \\
 & & & Y
\end{tikzcd}$$

We have a big diagram, but if we ignore all dotted lines, there is an obvious horizontal map $W \longrightarrow W$ to fill in the diagram, namely the id_W. Since the universal property guarantees that there is a **unique** way to fill it in, we find that $\varphi \circ f = \mathrm{id}_W$. Reversing the pacmen gives the other equality $f \circ \varphi = \mathrm{id}_{X \times Y}$. In other words, f is a homeomorphism, and $\varphi = f^{-1}$. ■

This argument may seem strange the first time you see it, but it is a typical argument that applies any time you define an object via a universal property. The argument shows that any two objects satisfying the universal property must be "the same".

Ok, so we understand $X \times Y$ as a topological space. What about a product of more than two spaces? Well, if we have a finite collection $X_1, \ldots, X_n$ of spaces, the product topology on $X_1 \times \cdots \times X_n$ has basis given by the $U_1 \times \cdots \times U_n$, or equivalently, subbasis given by the $p_i^{-1}(U_i)$. Note that this is equivalent because the basis element $U_1 \times \cdots \times U_n$, is a finite intersection of the subbasis elements $p_i^{-1}(U_i)$.

But what about the product of an *arbitrary* number of spaces? For instance, we might want to consider a countable infinite product $\mathbb{R} \times \ldots$. First, we stop to think about arbitrary products *as sets*.

Let X_i, for $i \in I$, be sets. The cartesian product $\prod_{i \in I} X_i$ is the collection of tuples (x_i), where $x_i \in X_i$. This means that for each $i \in I$, we want an element $x_i \in X_i$. In other words, we should have a function

$$x_{-} : I \longrightarrow X = \bigcup_i X_i$$

with the condition that this function satisfies $x_i \in X_i$. With this language, the "projection" $\prod_{i \in I} X_i \longrightarrow X_i$ is simply the restriction along $\{i\} \hookrightarrow I$.

In the case that all X_i are the same set X, then $\prod_i X_i$ is simply the set of functions $I \longrightarrow X$. So, the countably infinite product of $\mathbb{R}$ with itself is synonymous with the collection of sequences in $\mathbb{R}$.

Turning now to the question of a topology on $\prod_i X_i$, we can think about either the basis consisting or products $\prod_i U_i$, or we can think about the subbasis given by the $p_i^{-1}(U_i)$. But these are no longer equivalent! Which one is the "right" one?

Again, the product topology should be the universal example of a space with continuous maps to each X_i. So we want a minimal topology which makes the projection maps continuous. The topology given by the subbasis $p_i^{-1}(U_i)$ satisfies precisely this condition, but the one given by the basis $\{\prod_i U_i\}$ has more open sets.

Definition 10.4. Suppose given a collection of spaces X_i. The **product topology** on $\prod_i X_i$ is generated by the subbasis $p_i^{-1}(U_i)$. The **box topology** on $\prod_i X_i$ is generated by the basis $\{\prod_i U_i\}$.

As discussed above, the box topology has more open sets; in other words, the box topology is finer than the product topology. To see that the box topology does not have the universal property we want, consider the following example: let $\Delta : \mathbb{R} \longrightarrow \prod_{n \in \mathbb{N}} \mathbb{R}$ be the diagonal map, all of whose component maps are simply the identity. For each n, let $I_n = (\frac{-1}{n}, \frac{1}{n})$. In the box topology, the subset $I = \prod_n I_n \subseteq \prod_n \mathbb{R}$ is an open set, but

$$\Delta^{-1}(I) = \bigcap_n \mathrm{id}^{-1}(I_n) = \bigcap_n I_n = \{0\}$$

is not open. So the diagonal map is not continuous in the box topology!

We have established that we should prefer the product topology over the box topology, at least from a conceptual point of view, but we have only described this in terms of a subbasis, whereas we have a basis for the box topology. What is a basis for the product topology? By intersecting finitely many subbasis elements, we get the basis

$$\mathcal{B}_{\text{prod}} = \left\{ \prod_i U_i \mid U_i \subseteq X_i \text{ is open, and only finitely many } U_i \text{ are } \textbf{proper} \text{ subsets} \right\}.$$

In other words, we take boxes which are as large as possible in all but finitely many coordinates.