14. Mon, Sept. 30

Last time, we defined the quotient topology coming from a continuous surjection $q : X \longrightarrow Y$. Recall that q is a quotient map (and Y has the quotient topology) if $V \subseteq Y$ is open precisely when $q^{-1}(V) \subseteq X$ is open.

Example 14.1. (Collapsing a subspace) Let $A \subseteq X$ be a subspace. We define a relation on X as follows: $x \sim y$ if both are points in A or if neither is in A and $x = y$. Here, we have one equivalence class for the subset A, and every point outside of A is its own equivalence class. Standard notation for the set $X/\sim$ of equivalence classes under this relation is X/A. The universal property can be summed up as saying that any map on X which is constant on A factors through the quotient X/A.

For example, we considered last time the example $\mathbb{R}/(-\infty, 0] \cong [0, \infty)$.

Example 14.2. Consider $\partial I \subseteq I$. The exponential map $e : I \longrightarrow S^1$ is constant on ∂I, so we get an induced continuous map $\varphi : I/\partial I \longrightarrow S^1$, which is easily seen to be a bijection. In fact, it is a homeomorphism. Once we learn about compactness, it will be easy to see that this is a closed map.

We show instead that it is open. A basis for $I/\partial I$ is given by $q(a, b)$ with $0 < a < b < 1$ and by $q([0, a) \cup (b, 1])$ with again $0 < a < b < 1$. It is clear that both are taken to basis elements for the subspace topology on S^1. It follows that φ is a homeomorphism.

Example 14.3. Generalizing the previous example, for any closed ball $D^n \subseteq \mathbb{R}^{n+1}$, we can consider the quotient $D^n/\partial D^n$. On your homework this week, you are asked to provide a continuous bijection $D^n/\partial D^n \longrightarrow S^n$. Again, we will see later that this must be a homeomorphism $D^n/\partial D^n \cong S^n$.

Example 14.4. On S^n we impose the equivalence relation $\mathbf{x} \sim -\mathbf{x}$. The resulting quotient space is known as n-dimensional real projective space and is denoted $\mathbb{RP}^n$.

Consider the case $n = 1$. We have the hemisphere inclusion $I \hookrightarrow S^1$ given by $x \mapsto e^{ix\pi}$. Then the composition $I \hookrightarrow S^1 \twoheadrightarrow \mathbb{RP}^1$ is a quotient map that simply identifies the boundary ∂I to a point. In other words, this is example 14.1 from above, and we conclude that $\mathbb{RP}^1 \cong S^1$. However, the higher-dimensional versions of these spaces are certainly not homeomorphic.

Example 14.5. Consider S^{2n-1} as a subspace of $\mathbb{C}^n$. We then have the coordinate-wise multiplication by elements of $S^1 \cong U(1)$ on $\mathbb{C}^n$. This multiplication restricts to a multiplication on the subspace S^{2n-1}, and we impose an equivalence relation $(z_1, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n)$ for all $\lambda \in S^1$. The resulting quotient space is the complex projective space $\mathbb{CP}^n$.

Example 14.6. On $I \times I$, we impose the relation $(0, y) \sim (1, y)$ and also the relation $(x, 0) \sim (x, 1)$. The resulting quotient space is the torus $T^2 = S^1 \times S^1$. We recognize this as the product of two copies of example 14.1, but beware that in general a product of quotient maps need not be a quotient map. torus

We discussed last time the fact that a quotient map need not be open. Nevertheless, there is a class of open sets that are always carried to open sets.

Definition 14.7. Let $q : X \longrightarrow Y$ be a continuous surjection. We say a subset $A \subseteq X$ is **saturated** (with respect to q) if it is of the form $q^{-1}(V)$ for some subset $V \subseteq Y$.

It follows that A is saturated if and only if $q^{-1}(q(A)) = A$. Recall that a **fiber** of a map $q : X \longrightarrow Y$ is the preimage of a single point. Then another description is that A is saturated if and only if it contains all fibers that it meets.

Proposition 14.8. *A continuous surjection $q : X \twoheadrightarrow Y$ is a quotient map if and only if it takes saturated open sets to saturated open sets.*

Proof. Exercise. ∎

A number of the examples above have secretly been examples of a more general construction, namely the quotient under the action of a group.

Definition 14.9. A **topological group** is a based space (G, e) with a continuous multiplication $m : G \times G \longrightarrow G$ and inverse $i : G \longrightarrow G$ satisfying all of the usual axioms for a group.

Remark 14.10. Munkres requires all topological groups to satisfy the condition that points are closed. We will not make this restriction, though the examples we will consider will all satisfy this.

15. Wed, Oct. 2

Last time, we introduced the idea of a topological group, which is simultaneously a group and a space, where the multiplication and inverse are required to be continuous.

Example 15.1.
(1) Any group G can be considered as a topological group equipped with the discrete topology. For instance, we have the cyclic groups $\mathbb{Z}$ and $C_n = \mathbb{Z}/n\mathbb{Z}$.
(2) The real line $\mathbb{R}$ is a group under addition, This is a topological group because addition and multiplication by -1 are both continuous. Note that here $\mathbb{Z}$ is at the same time both a subspace and a subgroup. It is thus a topological subgroup.
(3) If we remove zero, we get the multiplicative group $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ of real numbers.
(4) Inside $\mathbb{R}^\times$, we have the subgroup $\{1, -1\}$ of order two.
(5) $\mathbb{R}^n$ is also a topological group under addition. In the case $n = 2$, we often think of this as $\mathbb{C}$.
(6) Again removing zero, we get the multiplicative group $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ of complex numbers.
(7) Inside $\mathbb{C}^\times$ we have the subgroup of complex numbers of norm 1, aka the circle group $S^1 \cong U(1) = SO(2)$.
(8) This last example suggests that matrix groups in general are good candidates. For instance, we have the topological group $Gl_n(\mathbb{R})$. This is a subspace of $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$. The determinant mapping $\det : M_n(\mathbb{R}) \longrightarrow \mathbb{R}$ is polynomial in the coefficients and therefore continuous. The general linear group is the complement of $\det^{-1}(0)$. It follows that $Gl_n(\mathbb{R})$ is an open subspace of $\mathbb{R}^{n^2}$.
(9) Inside $Gl_n(\mathbb{R})$, we have the closed subgroups $Sl_n(\mathbb{R})$, $O(n)$, $SO(n)$.

Let G be a topological group and fix some $h \in G$. Define $L_h : G \longrightarrow G$ by $L_h(g) = hg$. This is left multiplication by h. The definition of topological group implies that this is continuous, as L_h is the composition

$$G \xrightarrow{(h,\mathrm{id})} G \times G \xrightarrow{m} G.$$

Moreover, $L_{h^{-1}}$ is clearly inverse to L_h and continuous by the same argument, so we conclude that each L_h is a homeomorphism. Since $L_h(e) = h$, we conclude that neighborhoods around h look like neighborhoods around e. Since h was arbitrary, we conclude that neighborhoods around one point look like neighborhoods around any other point. This implies that a space like the unoin of the coordinate axes in $\mathbb{R}^2$ cannot be given the structure of topological group, as neighborhoods around the origin do not resemble neighborhoods around other points.

The main reason for studying topological groups is to consider their *actions* on spaces.

Definition 15.2. Let G be a topological group and X a space. A **left action** of G on X is a map $a : G \times X \longrightarrow X$ which is associative and unital. This means that $a(g, a(h, x)) = a(gh, x)$ and

$a(e, x) = x$. Diagrammatically, this is encoded as the following commutative diagrams

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{\mathrm{id}\times a} & G \times X \\ {\scriptstyle m\times\mathrm{id}}\downarrow & & \downarrow{\scriptstyle a} \\ G \times X & \xrightarrow[a]{} & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{e,\mathrm{id}} & G \times X \\ & {\scriptstyle \mathrm{id}}\searrow & \downarrow{\scriptstyle a} \\ & & X. \end{array}$$

It is common to write $g \cdot x$ or simply gx rather than $a(g, x)$.

Given an action of G on a space X, we define a relation on X by $x \sim y$ if $y = g \cdot x$ for some g. The equivalence classes are known as **orbits** of G in X, and the quotient of X by this relation is typically written as X/G. Really, the notation X/G should be reserved for the quotient by a *right action* of G on X, and the quotient by a left action should be $G\backslash X$.

Example 15.3.

(1) For any G, left multiplication gives an action of G on itself! This is a transitive action, meaning that there is only one orbit, and the quotient G/G is just a point.

Note that we saw above that, for each $h \in G$, the map $L_h : G \longrightarrow G$ is a homeomorphism. This generalizes to any action. For each $g \in G$, the map $a(g, -) : X \longrightarrow X$ is a homeomorphism.

(2) For any (topological) subgroup $H \leq G$, left multiplication by elements of H gives a left action of H on G. Note that an orbit here is precisely a right coset Hg. The quotient is $H\backslash G$, the set of right cosets of H in G.

(3) Consider the subgroup $\mathbb{Z} \leq \mathbb{R}$. Since $\mathbb{R}$ is abelian, we don't need to worry about about left vs. right actions or left vs. right cosets. We then have the quotient $\mathbb{R}/\mathbb{Z}$, which is again a topological group (again, $\mathbb{R}$ is abelian, so $\mathbb{Z}$ is normal).

What is this group? Once again, consider the exponential map $\exp : \mathbb{R} \longrightarrow S^1$ given by $\exp(x) = e^{2\pi i x}$. This is surjective, and it is a homomorphism since

$$\exp(x + y) = \exp(x)\exp(y).$$

The First Isomorphism Theorem in group theory tells us that $S^1 \cong \mathbb{R}/\ker(\exp)$, at least as a group. The kernel is precisely $\mathbb{Z} \leq \mathbb{R}$, and it follows that $S^1 \cong \mathbb{R}/\mathbb{Z}$ as a group. To see that this is also a homeomorphism, we need to know that $\exp : \mathbb{R} \longrightarrow S^1$ is a quotient map, but this follows from our earlier verification that $I \longrightarrow S^1$ is a quotient.

(4) Similarly, we can think of $\mathbb{Z}^n$ acting on $\mathbb{R}^n$, and the quotient is $\mathbb{R}^n/\mathbb{Z}^n \cong (S^1)^n = T^n$.

(5) The group $Gl(n)$ acts on $\mathbb{R}^n$ (just multiply a matrix with a vector), but this is not terribly interesting, as there are only two orbits: the origin is a closed orbit, and the complement is an open orbit. Thus the quotient space consists of a closed point and an open point.

(6) More interesting is the action of the subgroup $O(n)$ on $\mathbb{R}^n$. Using the fact that orthogonal matrices preserve norms, it is not difficult to see that the orbits are precisely the spheres around the origin. We claim that the quotient is the space $[0, \infty)$ (thought of as a subspace of $\mathbb{R}$).

To see this, consider the continuous surjection $|-| : \mathbb{R}^n \longrightarrow [0, \infty)$. By considering how this acts on open balls, you can show that this is an open map and therefore a quotient. But the fibers of this map are precisely the spheres, so it follows that this is the quotient induced by the above action of $O(n)$.

16. Fri, Oct. 4

At the end of class last time, we were looking at the example of $O(n)$ acting on $\mathbb{R}^n$, and we claimed that the quotient was $[0, \infty)$. We saw that the relation coming from the $O(n)$-action was the same as that coming from the surjection $\mathbb{R}^n \longrightarrow [0, \infty)$. Namely, we identify points if and only if they have the same norm. To see that the quotient by the $O(n)$-action is homeomorphic to $[0,\infty)$, it remains to show that the norm map $\mathbb{R}^n \longrightarrow [0,\infty)$ is a quotient map. We know already that it is a continuous surjection, and by considering basis elements (open balls) in $\mathbb{R}^n$, we can see that it is open as well. We leave this verification to the industrious student!

Why does the above argument show that the quotient $\mathbb{R}^n/O(n)$ is homeomorphic to $[0,\infty)$. We now have two quotient maps out of $\mathbb{R}^n$, and they are defined using the same equivalence relation on $\mathbb{R}^n$. By the universal property of quotients, the two spaces are homeomorphic!

Let's get on with more examples.

Example 16.1. (1) Let $\mathbb{R}^\times$ act on $\mathbb{R}^n$ via scalar multiplication. This action preserves lines, and within each line there are two orbits, one of which is the origin. Note that the only saturated open set containing 0 is $\mathbb{R}^n$, so the only neighborhood of 0 in the quotient is the entire space.

(2) Switching from n to $n+1$ for convenience, we can remove that troublesome 0 and let $\mathbb{R}^\times$ act on $X_{n+1} = \mathbb{R}^{n+1} \setminus \{0\}$. Here the orbits are precisely the lines (with origin removed). The quotient is $\mathbb{RP}^n$.

To see this, recall that we defined $\mathbb{RP}^n$ as the quotient of S^n by the relation $\mathbf{x} \sim -\mathbf{x}$. This is precisely the relation that arises from the action of the subgroup $C_2 = \{1,-1\} \leq \mathbb{R}^\times$ on $S^n \subseteq \mathbb{R}^{n+1}$.

Now notice that the map $\mathbb{R}^{n+1} \setminus \{0\} \longrightarrow S^n \times \mathbb{R}_{>0}$ given by $\mathbf{x} \mapsto \left(\frac{\mathbf{x}}{\|\mathbf{x}\|}, \|\mathbf{x}\|\right)$ is a homeomorphism. Next, note that we have an isomorphism $\mathbb{R}^\times \cong C_2 \times \mathbb{R}^\times_{>0}$. Thus the quotient $(\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^\times$ can be viewed as the two step quotient $\Big((S^{n-1} \times \mathbb{R}_{>0})/\mathbb{R}^\times_{>0})\Big)/C_2$. But $(\mathbb{R}^{n-1} \times \mathbb{R}_{>0})/\mathbb{R}^\times_{>0} \cong S^{n-1}$, so we are done.

We can think of $\mathbb{RP}^n$ in yet another way. Consider the following diagram:

$$\begin{array}{ccccc} D^n & \longrightarrow & S^n & \longrightarrow & \mathbb{R}^{n+1} \setminus \{0\} \\ \downarrow & & \downarrow & & \downarrow \\ D^n/\sim & \longrightarrow & S^n/C_2 & \longrightarrow & \mathbb{R}^{n+1} \setminus \{0\}/\mathbb{R}^\times \end{array}$$

The map $D^n \longrightarrow S^n$ is the inclusion of a hemisphere. The relation on D^n is the relation $\mathbf{x} \sim -\mathbf{x}$, but only allowed *on the boundary* ∂D^n. All maps on the bottom are continuous bijections, and again we will see later that they are necessarily homeomorphisms.

Note that the relation we imposed on D^n does *not* come from an action of C_2 on D^n. Let us write $C_2 = \langle \sigma \rangle$. We can try defining

$$\sigma \cdot \mathbf{x} = \begin{cases} \mathbf{x} & \mathbf{x} \in \mathrm{Int}(D^n) \\ -\mathbf{x} & \mathbf{x} \in \partial(D^n), \end{cases}$$

where here the interior and boundary are taken in S^n. But this is not continuous, as the convergent sequence

$$\left(\sqrt{1-\frac{1}{n}}, 0, \ldots, 0, \sqrt{\frac{1}{n}}\right) \to (1,0,\ldots,0)$$

is taken by σ to a convergent sequence, but the new limit is not $\sigma(1,0,\ldots,0) = (-1,0,\ldots,0)$.

(3) We have a similar story for $\mathbb{CP}^n$. There is an action of $\mathbb{C}^\times$ on $\mathbb{C}^{n+1} \setminus \{0\}$, and the orbits are the punctured complex lines. We claim that the quotient is $\mathbb{CP}^n$.

We defined $\mathbb{CP}^n$ as a quotient of an S^1-action on S^{2n+1}. We also have a homeomorphism $\mathbb{C}^{n+1} \setminus \{0\} \cong S^{2n+1} \times \mathbb{R}_{>0}$ and an isomorphism $\mathbb{C}^\times \cong S^1 \times \mathbb{R}^\times_{>0}$. We can then describe $\mathbb{CP}^n$

as the two-step quotient

$$\big(\mathbb{C}^{n+1} \setminus \{0\}\big)/\mathbb{C}^{\times} \cong \Big((S^{2n+1} \times \mathbb{R}_{>0})/\mathbb{R}_{>0}^{\times}\Big)/S^1 \cong S^{2n+1}/S^1 = \mathbb{CP}^n.$$