17. Mon, Oct. 7

We have been studying actions of topological groups on spaces, and the resulting quotient spaces X/G. But there is another way to think about this material. Suppose you have a set Y that you would like to topologize. One way to create a topology on Y is as follows. Pick a point $y_0 \in Y$. If there is a transitive action of some topological group G on Y, then the orbit-stabilizer theorem asserts that Y can be identified with G/H, where $H \leq G$ is the stabilizer subgroup consisting of all $h \in G$ such that $h \cdot y_0 = y_0$. But G/H is a topological space, so we define the topology on Y to be the one coming from the bijection $Y \cong G/H$.

Example 17.1. (Grasmannian) We saw that the projective spaces can be identified with the set of lines in $\mathbb{R}^n$ or $\mathbb{C}^n$, suitably topologized. We can similarly consider the set of k-dimensional linear subspaces in $\mathbb{R}^n$ (or $\mathbb{C}^n$). It is not clear how to topologize this set.

However, there is a natural action of $O(n)$ on the set of k-planes in $\mathbb{R}^n$. Namely, if $A \in O(n)$ is an orthogonal matrix and $V \subseteq \mathbb{R}^n$ is a k-dimensional subspace, then $A(V) \subseteq \mathbb{R}^n$ is another k-dimensional subspace. Furthermore, this action is transitive. To see this, it suffices to show that given any subspace V, there is a matrix taking the standard subspace $E_k = \text{Span}\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$ to V. Thus suppose $V = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a k-dimensional subspace with given orthogonal basis. This can be completed to an orthogonal basis of $\mathbb{R}^n$. Then if A is the orthogonal matrix with columns the $\mathbf{v}_i$, A takes the standard subspace E_k to V.

The stabilizer of E_k is the subgroup of orthogonal matrices that take E_k to E_k. Such matrices are block matrices, with an orthogonal $k \times k$ matrix in the upper left and an orthogonal $(n-k)\times(n-k)$ matrix in the lower right. In other words, the stabilizer subgroup is $O(k) \times O(n-k)$. It follows that the set of k-planes in $\mathbb{R}^n$ can be identified with the quotient

$$\text{Gr}_{k,n}(\mathbb{R}) = O(n)/O(k) \times O(n-k).$$

Note that, from this identification, it is clear that $\text{Gr}_{k,n} \cong \text{Gr}_{n-k,n}$. The map takes a k-plane in $\mathbb{R}^n$ to the orthogonal complement, which is an $n-k$-plane in $\mathbb{R}^n$. The corresponding map

$$O(n)/O(k) \times O(n-k) \longrightarrow O(n)/O(n-k) \times O(k)$$

is induced by a map $O(n) \longrightarrow O(n)$. This map on $O(n)$ is conjugation by a shuffle permutation that permutes k things past $n-k$ things.

There is an identical story for the complex Grasmannians, where $O(n)$ is replaced by $U(n)$.

Example 17.2. (Flag varieties) Continuing (why not?) in this vein, we can consider the sets of flags in $\mathbb{R}^n$ or $\mathbb{C}^n$. Recall that a flag is a chain of strict inclusions of linear subspaces $0 \leq V_1 \leq V_2 \leq \cdots \leq V_k = \mathbb{R}^n$. A flag is said to be **complete** if $\dim V_k = k$. The general linear group $\text{Gl}_n(\mathbb{R})$ acts transitively on the set of complete flags. Indeed, there is the standard complete flag $0 \leq E_1 \leq E_2 \leq \ldots$, where $E_k = \text{Span}\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}$, as above. Let $0 \leq V_1 \leq V_2 \leq \ldots$ be any other complete flag. Then if we choose a basis $\{\mathbf{v}_i\}$ for $\mathbb{R}^n$ such that $V_k = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, then it follows that the matrix A having the $\mathbf{v}_i$ for columns will take E_k to V_k.

In order to obtain a description of the complete flag variety $F(\mathbb{R}^n)$ as a space, we need to identify the stabilizer subgroup of a point. Let's look at the stabilizer of the standard complete flag. We want to know which matrices A will satisfy $A(\mathbf{e}_k) \in E_k$ for all k. The vector $A(\mathbf{e}_1)$ must be a nonzero multiple of $\mathbf{e}_1$, so the only nonzero entry in the first column of A is the top left entry. A similar analysis of the other columns shows that A must be upper-triangular (and nonsingular). If we thus denote the subgroup of upper-triangular matrices by B_n (B is for 'Borel'), then we see that the flag variety can be identified with the topological space

$$F(\mathbb{R}^n) \cong \text{Gl}_\text{n}(\mathbb{R})/B_n.$$

For some purposes, it is more convenient to work with the orthogonal group rather than the general linear group. This presents no real difficulty. We work with an orthonormal basis rather than any basis. Here we can see that the stabilizer of the standard flag consists of upper triangular orthogonal matrices, which coincides with the group of diagonal orthogonal matrices. These can only have 1 or -1 on the diagonal. We conclude that

$$F(\mathbb{R}^n) \cong O(n)/C_2 \times C_2 \times \cdots \times C_2.$$

There is a complex analogue as well. We have

$$F(\mathbb{C}^n) \cong \mathrm{Gl_n}(\mathbb{C})/B_n(\mathbb{C}) \cong U(n)/S^1 \times \cdots \times S^1 = U(n)/T^n.$$

What about non-complete flags? It is clear that if $\{V_i\}$ is a flag, then $\{AV_i\}$ will have the same "signature" (sequence of dimensions). But similar arguments to those above show that the general linear group or orthogonal group act transitively on the set of flags of a given signature, and we have

$$F(d_1, \dots, d_k; \mathbb{R}^n) \cong \mathrm{Gl_n}(\mathbb{R})/B_{n_1,\dots,n_k} \cong O(n)/O(n_1) \times O(n_2) \times \cdots \times O(n_k),$$

where $n_i = d_i - d_{i-1}$ and $B_{n_1,\dots,n_k}$ is the set of block-upper triangular matrices (with blocks of size n_1, n_2, etc.). Similarly,

$$F(d_1, \dots, d_k; \mathbb{C}^n) \cong \mathrm{Gl_n}(\mathbb{C})/B_{n_1,\dots,n_k}(\mathbb{C}) \cong U(n)/U(n_1) \times \cdots \times U(n_k).$$

18. Wed, Oct. 9

Exam day! I play on my laptop while your egos are crushed like tiny ants. Mwa-ha-ha...

19. Fri, Oct. 11

What we have done so far corresponds roughly to Chapters 2 & 3 of Lee. Now we turn to Chapter 4.

The first idea is connectedness. Essentially, we want to say that a space is cannot be decomposed into two disjoint pieces.

Definition 19.1. A **disconnection** (or separation) of a space X is a pair of disjoint, nonempty open subsets $U, V \subseteq X$ with $X = U \cup V$. We say that X is **connected** if it has no disconnection.

Example 19.2.
(1) If X is a discrete space (with at least two points), then any pair of disjoint nonempty subsets gives a disconnection of X.
(2) Let X be the subspace $(0, 1) \cup (2, 3)$ of $\mathbb{R}$. Then X is disconnected.
(3) More generally, if $X \cong A \coprod B$ for nonempty spaces A and B, then X is disconnected.
(4) Another example of a disconnected subspace of $\mathbb{R}$ is the subspace $\mathbb{Q}$. A disconnection of $\mathbb{Q}$ is given by $(-\infty, \pi) \cap \mathbb{Q}$ and $(\pi, \infty) \cap \mathbb{Q}$.
(5) Any set with the trivial topology is connected, since there is only one nonempty open set.
(6) Of the 29 topologies on $X = \{1, 2, 3\}$, 19 are connected, and the other 10 are disconnected. For example, the topology $\{\emptyset, \{1\}, X\}$ is connected, but $\{\emptyset, \{1\}, \{2, 3\}, X\}$ is not.
(7) If X is a space with the generic point (or included point) topology, in which the nonempty open sets are precisely the ones containing a special point x_0, then X is connected.
(8) If X is a space with the excluded point topology, in which the open proper subsets are the ones missing a special point x_0, then X is connected.
(9) The lower limit topology $\mathbb{R}_{\ell\ell}$ is disconnected, as the basis elements $[a, b)$ are both open and closed (clopen!), which means that their complements are open.

Proposition 19.3. *Let X be a space. The following are equivalent:*
(1) *X is disconnected*
(2) *$X \cong A \coprod B$ for nonempty spaces A and B*

(3) *There exists a nonempty, clopen, proper subset* $U \subseteq X$
(4) *There exists a continuous surjection* $X \twoheadrightarrow \{0,1\}$, *where* $\{0,1\}$ *has the discrete topology.*

Now let's look at an interesting example of a connected space.

Proposition 19.4. *The only (nonempty) connected subspaces of* $\mathbb{R}$ *are singletons and intervals.*

Proof. It is clear that singletons are connected. Note that, by an interval, we mean simply a convex subset of $\mathbb{R}$. It is clear that any connected subset must be an interval since if A is connected and $a < b < c$ with $a, c \in A$, then either $b \in A$ or $(-\infty, b) \cap A$ and $(b, \infty) \cap A$ give a separation of A.

So it remains to show that intervals are connected. Let $I \subseteq \mathbb{R}$ be an interval with at least two points, and let $U \subseteq I$ be nonempty and clopen. We wish to show that $U = I$. Let $a \in U$. We will show that $U \cap [a, \infty) = I \cap [a, \infty)$. A similar argument will show that $U \cap (-\infty, a] = I \cap (-\infty, a]$.

Consider the set

$$R_a = \{b \in I \mid [a, b] \subseteq U\}.$$

Note that $a \in R_a$, so that R_a is nonempty. If R_a is not bounded above, then $[a, \infty) \subseteq U \subseteq I$, and we have our conclusion. Otherwise, the set R_a has a supremum $s = \sup R_a$ in $\mathbb{R}$. Since we can express s as a limit of a U-sequence and since U is closed in I, it follows that if $s \in I$ then s must also lie in U.

Note that if $s \notin I$, then since I is an interval we have

$$U \cap [a, \infty) = [a, s) = I \cap [a, \infty).$$

On the other hand, as we just said, if $s \in I$ then $s \in U$. But U is open, so some ϵ-neighborhood of s (in I) lies in U. But no point in $(s, s + \epsilon/2)$ can lie in U (or I), since any such point would then also lie in R_a. Again, since I is an interval we have

$$U \cap [a, \infty) = [a, s] = I \cap [a, \infty).$$

∎

One of the most useful results about connected spaces is the following.

Proposition 19.5. *Let* $f : X \longrightarrow Y$ *be continuous. If* X *is connected, then so is* $f(X) \subseteq Y$.

Proof. This is a one-liner. Suppose that $U \subseteq f(X)$ is closed and open. Then $f^{-1}(U)$ must be closed and open, so it must be either $\emptyset$ or X. This forces $U = \emptyset$ or $U = f(X)$. ∎

Since the exponential map $\exp : [0, 1] \longrightarrow S^1$ is a continuous surjection, it follows that S^1 is connected. More generally, we have

Proposition 19.6. *Let* $q : X \longrightarrow Y$ *be a quotient map with* X *connected. Then* Y *is connected.*

Which of the other constructions we have seen preserve connectedness? All of them! (Well, except that subspaces of connected spaces need not be connected, as we have already seen.

Proposition 19.7. *Let* $A_i \subseteq X$ *be connected for each* i, *and assume that* $x_0 \in \bigcap_i A_i \neq \emptyset$. *Then* $\bigcup_i A_i$ *is connected.*

Proof. Assume each A_i is connected, and let $U \subseteq \bigcup_i A_i$ be nonempty and clopen. Then $x \in U$ for some $x \in \bigcup_i A_i$. Suppose $x \in A_{i_0}$. Then $U \cap A_{i_0}$ is nonempty and clopen in A_{i_0}, so $U \cap A_{i_0} = A_{i_0}$. In other words, $A_{i_0} \subseteq U$. Since $x_0 \in A_{i_0}$, it follows that $x_0 \in U$. But now for any other A_j, we have that $x_0 \in A_j \cap U$, so that $A_j \cap U$ is nonempty and clopen in A_j. It follows that $A_j \subseteq U$. ∎