23. Mon, Oct. 21

Like connectedness, compactness is preserved by continuous functions.

Proposition 23.1. *Let $f : X \longrightarrow Y$ be continuous, and assume that X is compact. Then $f(X)$ is compact.*

Proof. Let $\mathcal{V}$ be an open cover of $f(X)$. Then $\mathcal{U} = \{f^{-1}(V) \mid V \in \mathcal{V}\}$ is an open cover of X. Let $\{U_1, \dots, U_k\}$ be a finite subcover. It follows that the corresponding $\{V_1, \dots, V_k\}$ is a finite subcover of $\mathcal{V}$. ■

Example 23.2. Recall that we have the quotient map $\exp : [0,1] \longrightarrow S^1$. It follows that S^1 is compact.

Theorem 23.3 (Extreme Value Theorem)**.** *Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous. Then f attains a maximum and a minimum.*

Proof. Since f is continuous and $[a,b]$ is both connected and compact, the same must be true of its image. But the compact, connected subsets are precisely the closed intervals. ■

The following result is also quite useful.

Proposition 23.4. *Let X be Hausdorff and let $A \subseteq X$ be a compact subset. Then A is closed in X.*

Proof. Pick any point $x \in X \setminus A$ (if we can't, then $A = X$ and we are done). For each $a \in A$, we have disjoint neighborhoods $a \in U_a$ and $x \in V_a$. Since the U_a cover A, we only need finitely many, say $U_{a_1}, \dots, U_{a_k}$ to cover A. But then the intersection

$$V = V_{a_1} \cap \dots \cap V_{a_k}$$

of the corresponding V_a's is disjoint from the union of the U_a's and therefore also from A. Since V is a finite intersection of open sets, it is open and thus gives a neighborhood of x in $X \setminus A$. It follows that A is closed. ■

Exercise 23.5. *If $A \subseteq X$ is closed and X is compact, then A is compact.*

Combining these results gives the following long-awaited consequence.

Corollary 23.6. *Let $f : X \longrightarrow Y$ be continuous, where X is compact and Y is Hausdorff, then f is a closed map.*

In particular, if f is already known to be a continuous bijection, then it is automatically a homeomorphism. For example, this shows that the map $I/\partial I \longrightarrow S^1$ is a homeomorphism. Similarly, from Homework 5 we have $D^n/\partial D^n \cong S^n$.

We will next show that finite products of compact spaces are compact, but we first need a lemma.

Lemma 23.7 (Tube Lemma)**.** *Let X be compact and Y be any space. If $W \subseteq X \times Y$ is open and contains $X \times \{y\}$, then there is a neighborhood V of y with $X \times V \subseteq W$.*

Proof. For each $x \in X$, we can find a basic neighborhood $U_x \times V_x$ of (x,y) in W. The U_x's give an open cover of X, so we only need finitely many of them, say $U_{x_1}, \dots, U_{x_n}$. Then we may take $V = V_{x_1} \cap \dots \cap V_{x_n}$. ■

Proposition 23.8. *Let X and Y be nonempty. Then $X \times Y$ is compact if and only if X and Y are compact.*

Proof. As for connectedness, the continuous projections make X and Y compact if $X \times Y$ is compact.

Now suppose that X and Y are compact and let $\mathcal{U}$ be an open cover. For each $y \in Y$, the cover $\mathcal{U}$ of $X \times Y$ certainly covers the slice $X \times \{y\}$. This slice is homeomorphic to X and therefore finitely-covered by some $\mathcal{V} \subset \mathcal{U}$. By the Tube Lemma, there is a neighborhood V_y of y such that the tube $X \times V_y$ is covered by the same $\mathcal{V}$. Now the V_y's cover Y, so we only need finitely many of these to cover X. Since each tube is finitely covered by $\mathcal{U}$ and we can cover $X \times Y$ by finitely many tubes, it follows that $\mathcal{U}$ has a finite subcover. ■

Theorem 23.9 (Heine-Borel)**.** *A subset $A \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof. Suppose A is compact. Then A must be closed in $\mathbb{R}^n$ since $\mathbb{R}^n$ is Hausdorff. The subset A must also be bounded according to Homework problem VI.5.

On the other hand, suppose that A is closed and bounded in $\mathbb{R}^n$. Since A is bounded, it is contained in $[-k, k]^n$ for some $k > 0$. But this product of intervals is compact since each interval is compact. Now A is a closed subset of a compact space, so it is compact. ■

Again, we have shown that compactness interacts well with finite products, and we would like a similar result in the arbitrary product case. This is a major theorem, known as the Tychonoff theorem. First, we show the theorem does not hold with the box topology.

Example 23.10. Let $D = [-1, 1]$ and consider $D^{\mathbb{N}}$, equipped with the box topology. For each k, let

$$U_k = D^{\mathbb{N}} \cap \Big((-2^k, 2^k) \times (-2^{k-1}, 2^{k-1} \times \dots\Big).$$

So

$$U_1 = [-1, 1] \times (-1, 1) \times (-1/2, 1/2) \times (-1/4, 1/4) \times \dots$$

and

$$U_2 = [-1, 1] \times [-1, 1] \times (-1, 1) \times (-1/2, 1/2) \times \dots$$

Then $\mathcal{U} = \{U_k\}$ is an open cover with no finite subcover.

24. Wed, Oct. 23

It turns out that the Tychonoff Theorem is *equivalent* to the axiom of choice. We will thus use a form of the axiom of choice in order to prove it.

Zorn's Lemma. Let P be a partially-ordered set. If every linearly-ordered subset of P has an upper bound in P, then P contains at least one maximal element.

Theorem 24.1 (Tychonoff)**.** *Let $X_i \neq \emptyset$ for all $i \in \mathcal{I}$. Then $\prod_i X_i$ is compact if and only if each X_i is compact.*

Proof. As we have seen a number of times, the implication ($\Rightarrow$) is trivial.

We now show the contrapositive of ($\Leftarrow$). Thus assume that $X = \prod_i X_i$ is not compact. We wish to conclude that one of the X_i must be noncompact. By hypothesis, there exists an open cover $\mathcal{U}$ of X with no finite subcover.

We first deal with the following case.

Special case: *$\mathcal{U}$ is a cover by subbasis elements.*

For each $i \in \mathcal{I}$, let $\mathcal{U}_i$ be the collection

$$\mathcal{U}_i = \{V \subseteq X_i \text{ open } \mid p_i^{-1}(V) \in \mathcal{U}\}.$$

For some i, the collection $\mathcal{U}_i$ must cover X_i, since otherwise we could pick $x_i \in X_i$ for each i with x_i not in the union of $\mathcal{U}_i$. Then the element $(x_i) \in \prod_i X_i$ would not be in $\mathcal{U}$ since it cannot be in any $p_i^{-1}(V)$. Then $\mathcal{U}_i$ cannot have a finite subcover, since that would provide a corresponding subcover of $\mathcal{U}$. It follows that X_i is not compact.

It remains to show that we can always reduce to the cover-by-subbasis case.

Consider the collection $\mathcal{N}$ of open covers of X having no finite subcovers. By assumption, this set is nonempty, and we can partially order $\mathcal{N}$ by inclusion of covers. Furthermore, if $\{\mathcal{U}_\alpha\}$ is a linearly order subset of $\mathcal{N}$, then $\mathcal{U} = \bigcup_\alpha \mathcal{U}_\alpha$ is an open cover, and it cannot have a finite subcover since a finite subcover of $\mathcal{U}$ would be a finite subcover of one of the $\mathcal{U}_\alpha$. Thus $\mathcal{U}$ is an upper bound in $\mathcal{N}$ for $\{\mathcal{U}_\alpha\}$. By Zorn's Lemma, $\mathcal{N}$ has a maximal element $\mathcal{V}$.

Now let $\mathcal{S} \subseteq \mathcal{V}$ be the subcollection consisting of the subbasis elements in $\mathcal{V}$. We claim that $\mathcal{S}$ covers X. *Suppose not.* Thus let $x \in X$ such that x is not covered by $\mathcal{S}$. Then x must be in V for some $V \in \mathcal{V}$. By the definition of the product topology, x must have a basic open neighborhood in $B \subset V$. But any basic open set is a finite intersection of subbasic open sets, so $B = S_1 \cap \dots S_k$. If x is not covered by $\mathcal{S}$, then none of the S_i are in $\mathcal{S}$. Thus $\mathcal{V} \cup \{S_i\}$ is not in $\mathcal{N}$ by maximality of $\mathcal{V}$. In other words, $\mathcal{V} \cup \{S_i\}$ has a finite subcover $\{V_{i,1}, \dots, V_{i,n_i}, S_1\}$. Let us write

$$\hat{V}_i = V_{i,1} \cup \dots \cup V_{i,n_i}.$$

Now

$$X = \bigcap_i \Big(S_i \cup \hat{V}_i\Big) \subseteq \Big(\bigcap_i S_i\Big) \cup \Big(\bigcup_i \hat{V}_i\Big) \subseteq V \cup \Big(\bigcup_i \hat{V}_i\Big)$$

This shows that $\mathcal{V}$ has a finite subcover, which contradicts that $\mathcal{V} \in \mathcal{N}$. We thus conclude that $\mathcal{S}$ covers X using only subbasis elements.

But now by the argument at the beginning of the proof, $\mathcal{S}$, and therefore $\mathcal{V}$ as well, has a finite subcover. This is a contradiction. ■

As we said already, Tychonoff's theorem is equivalent to the axiom of choice (which is equivalent to Zorn's Lemma).

Theorem 24.2. *Tychonoff $\Rightarrow$ axiom of choice.*

Proof. This argument is quite a bit simplier than the other implication. Let $X_i \neq \emptyset$ for all $i \in \mathcal{I}$. We want to show that $X = \prod_i X_i \neq \emptyset$.

For each i, define $Y_i = X_i \cup \{\infty_i\}$, where $\infty_i \notin X_i$. We topologize Y_i such that the only nontrivial open sets are X_i and $\{\infty_i\}$. Now for each i, let $U_i = p_i^{-1}(\infty_i)$. The collection $\mathcal{U} = \{U_i\}$ gives a collection of open subsets of $Y = \prod_i Y_i$, and this collection covers Y if and only if $X = \emptyset$. Each Y_i is compact since it has only four open sets. Thus Y must be compact by the Tychonoff theorem. But no finite subcollection of $\mathcal{U}$ can cover Y. For example, $U_i \cup U_j$ does not cover Y since $a \in X_i$ and $b \in X_j$, then we can define $(y_i) \in Y \setminus (U_i \cup U_j)$ by

$$y_k = \begin{cases} a & k = i \\ b & k = j \\ \infty_k & k \neq i, j \end{cases}$$

The same kind of argument will work for any finite collection of U_i's. Since $\mathcal{U}$ has no finite subcover and Y is compact, $\mathcal{U}$ cannot cover Y, so that X must be nonempty. ■

25. Fri, Oct. 25

(Start with sketch proof of $S^1 \wedge S^1 \cong S^2$ from Homework V).

Closely related to compactness is the following notion.

Definition 25.1. We say that a space X is **sequentially compact** if every sequence in X has a convergent subsequence.

Example 25.2. The open interval $(0,1)$ is not sequentially compact because $\{1/n\}$ has no subsequence that converges in $(0,1)$. If we consider instead $[0,1]$, this example no longer works, and we will see that $[0,1]$ is indeed sequentially compact.

In general, there is no direct relation between compactness and sequential compactness.

Example 25.3. Consider $X = I^I$. By the Tychonoff theorem, X is compact. However, it is not sequentially compact. Let $f_n \in X$ be defined by $f_n(x) =$ the nth digit in the binary expansion of x. We claim that (f_n) has no convergent subsequence. Recall that convergence in X means *pointwise* convergence of functions. Let (f_{n_k}) be any subsequence. In order for this to converge, it the sequence $f_{n_k}(x)$ would need to converge for every x. This is simply a sequence of 0's and 1's, so it must be eventually constant. But no matter the subsequence f_{n_k}, we can find an $x \in I$ whose corresponding sequence of digits is not eventually constant.

Example 25.4. Let

$$X = \left\{ x \in \prod_{\mathbb{R}} \{0,1\} \,\middle|\, x^{-1}(1) \text{ is countable.} \right\}$$

We here consider $\{0,1\}$ with the discrete topology, and X is a subspace of the product. For each $r \in \mathbb{R}$, let $B_r = \{x \in X \mid x(r) = 0\}$. This is a subbasis element and so is open. Then the collection $\{B_r\}_{r\in\mathbb{R}}$ gives an open cover of X, but it clearly has no finite subcover.

Now let (x_n) be a sequence in X. Let

$$S = \bigcup_n x_n^{-1}(1).$$

S is a countable union of countable sets, so it is countable. Let $Y = \prod_S \{0,1\}$, and let $q : X \longrightarrow Y$ be the restriction along $S \hookrightarrow \mathbb{R}$. Then $q(x_n)$ is a sequence in $Y = \{0,1\}^S$. It can be seen directly that Y is sequentially compact, so that some subsequence $q(x_{n_k})$ of $q(x_n)$ must converge to, say $y \in Y$. Let $z \in X$ be the function with $x^{-1}(1) = y^{-1}(1)$. But then x_{n_k} converges to z since each x_n is identically 0 on $\mathbb{R} \setminus S$.

We have shown that X is sequentially compact spaces but not compact.

Theorem 25.5. *If X is a metric space, then X is compact if and only if it is sequentually compact.*

Proof. See Munkres, Theorem 28.2 or Lee, Lemmas 4.42-4.44. ∎

In $\mathbb{R}^n$, this result is knwon by the following name.

Theorem 25.6 (Bolzano-Weierstrass). *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*