Math 551 - Topology I
Homework 7
Fall 2014

1. Given based spaces (X, x_0) and (Y, y_0), there is a natural axes inclusion $X \vee Y \hookrightarrow X \times Y$. Define the **smash product** of X and Y to be

$$X \wedge Y = (X \times Y)/(X \vee Y).$$

Show that there is a homeomorphism $S^1 \wedge S^1 \cong S^2$ or that more generally $S^n \wedge S^1 \cong S^{n+1}$. (Hint: Feel free to assume the existence of a homeomorphism $D^n \cong I^n$ that takes the boundary to the boundary.)

2. (Cantor set) Let $A_0 = I = [0,1]$. Define $A_1 = A_0 \setminus (\frac{1}{3}, \frac{2}{3})$. Similarly, define A_2 by removing the middle thirds of the intervals in A_1:

$$A_2 = A_1 \setminus \left(\left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \right).$$

In general, given A_n constructed in this way, we define A_{n+1} by removing the middle thirds of all intervals in A_n. Define the Cantor set to be

$$C = \bigcap_n A_n \subseteq [0,1].$$

   (a) Show that C is compact (without using part (d)).
   (b) Show that any compact, locally connected space has finitely many components. Conclude that C is not locally connected.
   (c) Show that C is totally disconnected (every connected component is a singleton).
   (d) Let $D = \{0,2\}$ with the discrete topology. Show that $C \cong \prod_n D$. (Hint: instead of binary expansions, think about ternary expansions of numbers in $[0,1]$.)

3. Let X be Hausdorff, and suppose that $C, D \subseteq X$ are disjoint compact subsets. Show that there are disjoint open sets $U, V \subseteq X$ with $C \subseteq U$ and $D \subseteq V$.

4. Let $p : X \longrightarrow Y$ be a closed, continuous, surjective map such that each fiber $p^{-1}(y)$ is compact. Show that if Y is compact, then so is X. (Hint: if U is open and contains $p^{-1}(y)$, show there is a neighborhood V of y such that $p^{-1}(V) \subseteq U$.)