Math 551 - Topology I
Homework 8
Fall 2014

1. (Metric completion) Let X be a metric space. Let C_X be the set of Cauchy sequences in X. Define a relation on C_X by

$$(x_n) \sim (y_n) \quad \text{if} \quad \lim_n \mathrm{d}_X(x_n, y_n) = 0.$$

Define $X^* = (C_X/\sim)$. Then $\mathrm{d}\Big((x_n),(y_n)\Big) = \lim_n \mathrm{d}_X(x_n, y_n)$ defines a metric on X^*, and the map

$$\begin{aligned} \iota : X &\longrightarrow X^* \\ x &\mapsto (x, x, \dots) \end{aligned}$$

is an isometric embedding.

   (a) Show that $\iota(X)$ is dense in X^*.

   (b) Show that if A is a dense subset of a metric space Z and every Cauchy sequence in A converges in Z, then Z is complete. Conclude that X^* is complete.

   (c) Show that X is totally bounded if and only if X^* is compact.

2. (a) Find an example of a bijective continuous map $f : X \longrightarrow Y$, where X is locally compact but Y is not.

   (b) Show that if $f : X \longrightarrow Y$ is a continuous, open surjection and X is locally compact, then Y must be locally compact.

3. Let $X = \mathbb{R} \times \mathbb{Z}/\sim$, where $\sim$ is the equivalence relation generated by $(x, n) \sim (x, k)$ for all $n, k \in \mathbb{Z}$ and $x \neq 0$. Show that X is locally compact but does not have a basis of precompact open sets.

4. (Stereographic Projection) Let $N = (0, \dots, 0, 1) \in S^n$ be the North Pole. Define a homeomorphism $S^n \setminus \{N\} \cong \mathbb{R}^n$ as follows. For each $x \neq N \in S^n$, consider the ray starting at N and passing through x. This meets the equatorial hyperplane (defined by $x_{n+1} = 0$) in a point, which we call $p(x)$.

   (a) Determine a formula for p and show that it gives a homeomorphism.

   (b) Conclude that the one-point compactification of $\mathbb{R}^n$ is S^n.