32. Mon, Nov. 10

Last time, we saw that a space is normal if and only if any two closed sets can be separated by a continuous function (modulo the T_1 condition). Here is another important application of normal spaces.

Theorem 32.1 (Tietze extension theorem)**.** *Suppose X is normal and $A \subseteq X$ is closed. Then any continuous function $f : A \longrightarrow [0,1]$ can be extended to a continuous function $\tilde{f} : X \longrightarrow [0,1]$.*

Again, this becomes an if and only if if we drop the T_1-condition from normal.

It is also easy to see that the result fails if we drop the hypothesis that A be closed. Consider $X = S^1$ and A is the complement of a point. Then we know that $A \cong (0,1)$, but this homeomorphism cannot extend to a map $S^1 \twoheadrightarrow (0,1)$.

Sketch of proof. It is more convenient for the purpose of the proof to work with the interval $[-1,1]$ rather than $[0,1]$. Thus suppose $f : A \longrightarrow [-1,1]$ is continuous. Then $A_1 = f^{-1}([-1,-1/3])$ and $B_1 = f^{-1}([1/3,1])$ are closed, disjoint subsets of A and therefore also of X. Since X is normal, we have a Urysohn function $g_1 : X \longrightarrow [-1/3,1/3]$ which separates A_1 and B_1. It is simple to check that $|f(a) - g_1(a)| \leq 2/3$ for all $a \in A$. In other words, we have a map

$$f_1 = f - g_1 : A \longrightarrow [-2/3, 2/3].$$

Define $A_2 = f_1^{-1}([-2/3,-2/9])$ and $B_2 = f_1^{-1}([2/9,2/3])$. We get a Urysohn function $g_2 : X \longrightarrow [-2/9,2/9]$ which separates A_2 and B_2. Then the difference $f_2 = f - g_1 - g_2$ maps to $[-4/9,4/9]$. We continue in this way, and in the end, we get a sequence of functions (g_n) defined on X, and we define $g = \sum_n g_n$. By construction, this agrees with f on A (the difference will be less than $(2/3)^n$ for all n). The work remains in showing that the series defining g converges (compare to a geometric series) and that the resulting g is continuous (show that the series converges uniformly). See [Munkres, Thm 35.1] for more details. ∎

Theorem 32.2 (Stone-Čech compactification)**.** *Suppose X is normal. There exists a "universal" compactification $\iota : X \longrightarrow Y$ of X, such that if $j : X \longrightarrow Z$ is any map to a compact Hausdorff space (for example a compactification), there is a unique map $q : Y \longrightarrow Z$ with $q \circ \iota = j$.*

$$\begin{tikzcd}
& Y \arrow[dr, dashed, "q"] & \\
X \arrow[ur, hook, "\iota"] \arrow[rr, "j"'] & & Z
\end{tikzcd}$$

33. Wed, Nov. 12

Proof. Given the space X, let $\mathcal{F} = \{ \text{ cts } f : X \longrightarrow [0,1]\}$. Define

$$\iota : X \longrightarrow [0,1]^{\mathcal{F}}$$

by $\iota(x)_f = f(x)$. This is continuous because each coordinate function is given by some $f \in \mathcal{F}$. The infinite cube is compact Hausdorff, and we let $Y = \overline{\iota(X)}$. It remains to show that ι is an embedding and also to demonstrate the universal property.

First, ι is injective by Urysohn's lemma: given distinct points x and y in X, there is a Urysohn function separating x and y, so $\iota(x) \neq \iota(y)$.

Now suppose that $U \subseteq X$ is open. We wish to show that $\iota(U)$ is open in $\iota(X)$. Pick $x_0 \in U$. Again by Urysohn's lemma, we have a function $g : X \longrightarrow [0,1]$ with $g(x_0) = 0$ and $g \equiv 1$ outside of U. Let

$$B = \{\iota(x) \in \iota(X) \mid g(x) \neq 1\} = \iota(X) \cap p_g^{-1}([0,1)).$$

Certainly $\iota(x_0) \in B$. Finally, $B \subset \iota(U)$ since if $\iota(x) \in B$, then $g(x) \neq 1$. But $g \equiv 1$ outside of U, so x must be in U.

For the universal property, suppose that $j : X \longrightarrow Z$ is a map to a compact Hausdorff space. Then Z is also normal, and the argument above shows that it embeds inside some large cube $[0,1]^K$. For each $k : Z \longrightarrow [0,1]$ in K, we thus get a coordinate map $i_k = p_k \circ j : X \longrightarrow [0,1]$, and it is clear how to extend this to get a map $q_k : Y \longrightarrow [0,1]$: just take q_k to be the projection map p_{i_k} onto the factor labelled by the map i_k. Piecing these together gives a map $q : Y \longrightarrow [0,1]^K$, but it restricts to the map j on the subset X. Since j has image in the closed subset Z, it follows that $q(Y) \subseteq Z$ since q is continuous and $\iota(X)$ is dense in Y. Note that q is the *unique* extension of j to Y since Z is Hausdorff and $\iota(X)$ is dense in Y. ■

Corollary 33.1. *Suppose that X is normal, and that $X \hookrightarrow Z$ is any compactification. Then Z is a quotient of the Stone-Čech compactification Y of X.*

Proof. According to the Theorem 32.2, we have a continuous map $q : Y \longrightarrow Z$ whose restriction to X is the given map $j : X \hookrightarrow Z$. The map q is closed since Y is compact and Z is Hausdorff. Also, $j(X)$ is dense in Z, and $j(X) = q(\iota(X)) \subseteq q(Y)$ so $q(Y) = Z$. In other words, q is closed, continuous, and surjective, therefore it is a quotient map. ■

The Stone-Čech compactification has consequences for *metrizability* of a space. Consider first the case that the index set J is countable.

Proposition 33.2. *Let Y be a metric space, and let $\overline{d} : Y \times Y \longrightarrow \mathbb{R}$ be the associated truncated metric. Then the formula*

$$D(\mathbf{y}, \mathbf{z}) = \sup \left\{ \frac{\overline{d}(y_n, z_n)}{n} \right\}$$

defines a metric on $Y^{\mathbb{N}}$, and the induced topology is the product topology.

Proof. We leave as an exercise the verification that this is a metric. We check the statement about the topology. For each n, let $p_n : Y^{\mathbb{N}} \longrightarrow Y$ be evaluation in the nth place. This is continuous, as given $\mathbf{y} \in Y^{\mathbb{N}}$ and $\epsilon > 0$, we let $\delta = \epsilon/n$. Then if $D(\mathbf{y}, \mathbf{z}) < \delta$, it follows that

$$d(y_n, z_n) = n \frac{d(y_n, z_n)}{n} \leq n D(\mathbf{y}, \mathbf{z}) < n\delta = \epsilon.$$

By the universal property of the product, we get a continuous bijection $p : Y^{\mathbb{N}} \longrightarrow \prod_{\mathbb{N}} Y$.

It remains to show that p is open. Thus let $B \subseteq Y^{\mathbb{N}}$ be an open ball, and let $\mathbf{y} \in p(B) = B$. We want to find a basis element U in the product topology with $\mathbf{y} \in U \subseteq B$. For convenience, we replace B by $B_\epsilon(\mathbf{y})$ for small enough ϵ. Take N large such that $1/N < \epsilon$. Then define

$$U = \bigcap_{i=1}^{N} p_i^{-1}(B_\epsilon(y_i)).$$

Let $\mathbf{z} \in Y^{\mathbb{N}}$. Recall that we have truncated our metric on Y at 1. Thus if $n > N$, we have that $\overline{d}(y_n, z_n)/n \leq 1/n \leq 1/N < \epsilon$. It follows that for any $\mathbf{z} \in U$, we have $\mathbf{z} \in B_\epsilon(\mathbf{x})$ as desired. ■

34. Fri, Nov. 14

On the other hand, if J is uncountable, then $[0,1]^J$ need not be metric, as the following example shows.

Example 34.1. The sequence lemma fails in $\mathbb{R}^{\mathbb{R}}$. Let $A \subseteq \mathbb{R}^{\mathbb{R}}$ be the subset consisting of functions that zero at all but finitely many points. Let g be the constant function at 1. Then $g \in \overline{A}$, since if

$$U = \bigcap_{x_1, \ldots, x_k} p_{x_i}^{-1}(a_i, b_i)$$

is a neighborhood of g, then the function

$$f(x) = \begin{cases} 1 & x \in \{x_1, \ldots, x_k\} \\ 0 & \text{else} \end{cases}$$

is in $U \cap A$. But no sequence in A can converge to g (recall that convergence in the product topology means pointwise convergence). For suppose f_n is a sequence in A. For each n, let $Z_n = \operatorname{supp}(f_n)$ (the support is the set where f_n is nonzero). Then the set

$$\mathcal{Z} = \bigcup_n Z_n$$

is countable, and on the complement of $\mathcal{Z}$, all f_n's are zero. So it follows that the same must be true for any limit of f_n. Thus the f_n cannot converge to g.

This finally leads to a characterization of those topological spaces which come from metric spaces.

Theorem 34.2. *If X is normal and second countable, then it is metrizable.*

Proof. Since X is normal, we can embed X as above inside a cube $[0,1]^J$ for some J. Above, we took J to be the collection of all functions $X \longrightarrow [0,1]$.

To get a countable indexing set J, start with a countable basis $\mathcal{B} = \{B_n\}$ for X. For each pair of indices n, m for which $\overline{B}_n \subset B_m$, the Urysohn lemma gives us a function $g_{n,m}$ vanishing on $\overline{B}_n$ and equal to 1 outside B_m. We take $J = \{g_{n,m}\}$. Going back to the proof of the Stone-Čech-compactification, we needed, for any $x_0 \in X$ and $x_0 \in U$, to be able to find a function vanishing at x_0 but equal to 1 outside of U.

Take a basis element B_m satisfying $x_0 \in B_m \subset U$. Since X is normal, we can find an open set V with $x_0 \in V \subset \overline{V} \subset B_m$. Find a B_n inside of V, and we are now done: namely, the function $g_{n,m}$ is what we were after. ■

We now come back to a result that we previously put off.

Theorem 34.3. *Suppose X is locally compact, Hausdorff, and second-countable. Then X is normal.*

Proof. Given closed, disjoint subsets A and B, we want to separate them using disjoint open sets.

Consider first the case where $A = \{a\}$ is a point. Writing $V = X \setminus B$, we have $a \in V$, and we wish to find U with $a \in U \subseteq \overline{U} \subseteq V$. Since X is locally compact, Hausdorff, we can consider the one-point compactificaiton $\widehat{X}$. But now we have $a \in V \subseteq \widehat{X}$, and $\widehat{X}$ is compact Hausdorff and therefore normal. So we get the desired U. Note that the same argument does not work for a general A, since we would not know that A is closed in $\widehat{X}$ (unless A is compact). We have proved that X is regular (T_3).

Now let A and B be general closed, disjoint subsets. For each $a \in A$, we can find a basis element U_a with $a \in U_a \subseteq \overline{U_a} \subset X \setminus B$. Since our basis is countable, we can enumerate all such U_a's to get a countable cover $\{U_n\}$ of A which is disjoint from B. Similarly, we get a countable cover $\{V_n\}$ of B which is disjoint from A. But the U_n's need not be disjoint from the V_k's so we need to fix this.

Define new covers of A and B, respectively, as follows. For each n, define

$$\widehat{U}_n = U_n \setminus \bigcup_{j=1}^{n} \overline{V}_j \qquad \text{and} \qquad \widehat{V}_n = V_n \setminus \bigcup_{j=1}^{n} \overline{U}_j$$

The $\widehat{U}_n$'s still cover A because we have removed the $\overline{V}_j$, which were all disjoint from A. Similarly, the $\widehat{V}_n$ cover B. Moreover, $\widehat{U}_n$ is disjoint from $\widehat{V}_j$ because, assuming WLOG that $n < j$, the closure of U_n has been removed from V_j in the formation of $\overline{V}_j$. ■

Combining the previous results gives

Corollary 34.4. *Suppose X is locally compact, Hausdorff, and second-countable. Then X is metrizable.*