42. Mon, Dec. 8

Last time, we were discussing CW complexes, and we considered two different CW structures on S^n. We continue with more examples.

(2) $\mathbb{RP}^n$. Let's start with $\mathbb{RP}^2$. Recall that one model for this space was as the quotient of D^2, where we imposed the relation $x \sim -x$ on the boundary. If we restrict our attention to the boundary S^1, then the resulting quotient is $\mathbb{RP}^1$, which is again a circle. The quotient map $q : S^1 \longrightarrow S^1$ is the map that winds twice around the circle. In complex coordinates, this would be $z \mapsto z^2$. The above says that we can represent $\mathbb{RP}^2$ as the pushout

$$\begin{tikzcd} S^1 \arrow[r, "\iota"] \arrow[d, "q"'] & D^2 \arrow[d] \\ S^1 \arrow[r] & \mathbb{RP}^2 \end{tikzcd}$$

If we build the 1-skeleton S^1 using a single 0-cell and a single 1-cell, then $\mathbb{RP}^2$ has a single cell in dimensions ≤ 2.

More generally, we can define $\mathbb{RP}^n$ as a quotient of D^n by the relation $x \sim -x$ on the boundary S^{n-1}. This quotient space of the boundary was our original definition of $\mathbb{RP}^{n-1}$. It follows that we can describe $\mathbb{RP}^n$ as the pushout

$$\begin{tikzcd} S^{n-1} \arrow[r, "\iota"] \arrow[d, "q"'] & D^n \arrow[d] \\ \mathbb{RP}^{n-1} \arrow[r] & \mathbb{RP}^n \end{tikzcd}$$

Thus $\mathbb{RP}^n$ can be built as a CW complex with a single cell in each dimension $\leq n$.

(3) $\mathbb{CP}^n$. Recall that $\mathbb{CP}^1 \cong S^2$. We can think of this as having a single 0-cell and a single 2-cell. We defined $\mathbb{CP}^2$ as the quotient of S^3 by an action of S^1 (thought of as $U(1)$). Let $\eta : S^3 \longrightarrow \mathbb{CP}^1$ be the quotient map. What space do we get by attaching a 4-cell to $\mathbb{CP}^1$ by the map η? Well, the map η is a quotient, so the pushout $\mathbb{CP}^1 \cup_\eta D^4$ is a quotient of D^4 by the S^1-action on the boundary.

Now include D^4 into $S^5 \subseteq \mathbb{C}^3$ via the map

$$\varphi(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4, \sqrt{1 - \sum x_i^2}, 0).$$

(This would be a hemi-equator.) We have the diagonal $U(1)$ action on S^5. But since any nonzero complex number can be rotated onto the positive x-axis, the image of φ meets every S^1-orbit in S^5, and this inclusion induces a homeomorphism on orbit spaces

$$D^4/U(1) \quad \cong S^5/U(1) = \mathbb{CP}^2.$$

We have shown that $\mathbb{CP}^2$ has a cell structure with a single 0-cell, 2-cell, and 4-cell.

This story of course generalizes to show that any $\mathbb{CP}^n$ can be built as a CW complex having a cell in each even dimension.

43. Wednesday, Dec. 10

(4) (Torus) In general, a product of two CW complexes becomes a CW complex. We will describe this in the case $S^1 \times S^1$, where S^1 is built using a single 0-cell and single 1-cell.

Start with a single 0-cell, and attach two 1-cells. This gives $S^1 \vee S^1$. Now attach a single 2-cell to the 1-skeleton via the attaching map ψ defined as follows. Let us refer to the two circles in $S^1 \vee S^1$ as ℓ and r. We then specify $\psi : S^1 \longrightarrow S^1 \vee S^1$ by $\ell r \ell^{-1} r^{-1}$. What we mean is to trace out ℓ on the first quarter of the domain, to trace out r on the second quarter, to run ℓ in reverse on the third quarter, and finally to run r in reverse on the final quarter.

We claim that the resulting CW complex X is the torus. Since the attaching map $\psi : S^1 \longrightarrow S^1 \vee S^1$ is surjective, so is $\iota_{D^2} : D^2 \longrightarrow X$. Even better, it is a quotient map. On the other hand, we also have a quotient map $I^2 \longrightarrow T^2$, and using the homeomorphism $I^2 \cong D^2$ from before, we can see that the quotient relation in the two cases agrees. We say that this homeomorphism $T^2 \cong X$ puts a cell structure on the torus. There is a single 0-cell (a vertex), two 1-cells (the two circles in $S^1 \vee S^1$), and a single 2-cell.

Let's talk about some of the (nice!) topological properties of CW complexes.

Lemma 43.1. *Let $\mathcal{E} = \{e_i^{n_i}\}$ be the set of all cells in X. Then X is a quotient of $\coprod_{\mathcal{E}} D^{n_i}$. In particular, $A \subseteq X$ is open (or closed) if and only if, for each cell i and corresponding characteristic map $\Phi_i : D^{n_i} \longrightarrow X$, the preimage $\Phi_i^{-1}(A)$ is open (or closed) in D^{n_i}.*

Proof. The forward implication is clear by continuity of the φ_i. For the other direction, suppose that each $\Phi_i^{-1}(A)$ is open. Then $A \cap X^0$ is open in X^0, since X^0 is just the disjoint union of its cells. Now assume by induction that $A \cap X^{n-1}$ is open in X^{n-1}. But, by the construction of the pushout, the n-skeleton X^n is a quotient of $X^{n-1} \amalg \coprod D^n$. Since $A \cap X^n$ pulls back to an open set in each piece of this coproduct, it must be open in $A \cap X^n$ by the definition of the quotient topology. Now, since $A \cap X^n$ is open in X^n for all n, A is open in X by property W. ■

Theorem 43.2. *Any CW complex X is normal.*

Proof. First, X is T_1 by the Lemma since any point obviously pulls back to a closed subset of every D_i^n. Let A and B be disjoint closed sets in X. We will show that X is normal by building a Urysohn function $f : X \longrightarrow [0,1]$ with $f(A) \equiv 0$ and $f(B) \equiv 1$. Because X satisfies property W, a function f defined on X is continuous if and only if its restriction to each X^n is continuous. We thus build the function f by building its restrictions f^n to X^n.

On X^0, we define

$$f^0(x) = \begin{cases} 0 & x \in A \cap X^0 \\ 1 & x \in B \cap X^0 \\ 1/2 & \text{else.} \end{cases}$$

Since X^0 is discrete, this is automatically continuous.

Now assume by induction that we have $f^{n-1} : X^{n-1} \longrightarrow [0,1]$ continuous with $f^{n-1}(A \cap X^{n-1}) \equiv 0$ and $f^{n-1}(B \cap X^{n-1}) \equiv 1$. Since we have a pushout diagram

$$\begin{tikzcd} \coprod S^{n-1} \arrow[r] \arrow[d] & \coprod D^n \arrow[d] \\ X^{n-1} \arrow[r] & X^n, \end{tikzcd}$$

by the universal property of the pushout, to define f^n on X^n, we need only specify a compatible pair of functions on X^{n-1} and on the disjoint union. On X^{n-1}, we take f^{n-1}. To define a map out of $\coprod D^n$, it is enough to define a map on each D^n.

For each n-cell e^i, define $W_i \subseteq D^n$ closed by $W_i = \partial D^n \cup \Phi_i^{-1}(A \cap X^n) \cup \Phi_i^{-1}(B \cap X^n)$. Define $g : W_i \longrightarrow [0,1]$ by

$$g(x) = \begin{cases} f^{n-1}(\varphi(x)) & x \in \partial D^n \\ 0 & x \in \Phi_i^{-1}(A \cap X^n) \\ 1 & x \in \Phi_i^{-1}(B \cap X^n). \end{cases}$$

We know that D^n is compact Hausdorff (or metric) and thus normal. Thus, by the Tietze extneion theorem (32.1) there is a Urysohn function for the disjoint closed sets $\Phi_i^{-1}(A \cap X^n)$ and $\Phi_i^{-1}(B \cap X^n)$ whose restriction to ∂D^n agrees with $f^{n-1} \circ \varphi_i$. Putting all of this together gives a Urysohn function on X^n for the $A \cap X^n$ and $B \cap X^n$. By induction, we are done. ■

44. Fri, Dec. 12

Even better,

Theorem 44.1 (Lee, Theorem 5.22)**.** *Every CW complex is paracompact.*

Proposition 44.2. *Any CW complex X is locally path-connected.*

Proof. Let $x \in X$ and let U be any open neighborhood of x. We want to find a path-connected neighborhood V of x in U. Recall that a subset $V \subseteq X$ is open if and only if $V \cap X^n$ is open for all n. We will define V by specifying open subsets $V^n \subseteq X^n$ with $V^{n+1} \cap X^n = V^n$ and then setting $V = \cup V^n$.

Suppose that x is contained in the cell e_i^n. We set $V^k = \emptyset$ for $k < n$. We specify V_n by defining $\Phi_j^{-1}(V^n)$ for each n-cell e_j^n. If $j \neq i$, we set $\Phi_j^{-1}(V_n) = \emptyset$. We define $\Phi_i^{-1}(V_n)$ to be an open n-disc around $\Phi_i^{-1}(x)$ whose closure is contained in $\Phi_i^{-1}(U)$. Now suppose we have defined V^k for some $k \geq n$. Again, we define V^{k+1} by defining each $\Phi_i^{-1}(V^{k+1})$. By assumption, $\overline{\Phi_i^{-1}(V^k)} \subseteq \partial D^{k+1} \subseteq \Phi_i^{-1}(U)$. By the Tube lemma, there is an $\epsilon > 0$ such that (using radial coordinates) $\Phi_i^{-1}(V^k) \times (1-\epsilon, 1] \subset U$. We define

$$\Phi_i^{-1}(V^{k+1}) = \Phi_i^{-1}(V^k) \times [1, 1-\epsilon/2),$$

which is path-connected by induction. This also guarantees that $\overline{V^{k+1}} \subset U \cap X^{k+1}$, allowing the induction to proceed. ■

Proposition 44.3 (Hatcher, A.1)**.** *Any compact subset K of a CW complex X meets finitely many cells.*

Proof. For each cell e_i meeting K, pick a point $k_i \in K \cap e_i$. Let $S = \{k_i\}$. We use proprety W to show that S is closed in X. It is clear that $S \cap X^0$ is closed in X^0 since X^0 is discrete. Assume that $S \cap X^{n-1}$ is closed in X^{n-1}. Now in X^n, the set $S \cap X^n$ is the union of the closed subset $S \cap X^{n-1}$ and the points k_i that lie in open n-cells. By Lemma 43.1, this set of k_i is closed as well.

The argument above in fact shows that any subset of S is closed, so that S is discrete. But S is closed in K, so S is compact. Since S is both discrete and compact, it must be finite. ■

Corollary 44.4. *Any CW complex has the closure-finite property, meaning that the closure of any cell meets finitely many cells.*

Proof. The closure of e_i is $\Phi_i(D_i^{n_i})$, which is compact. The result follows from the proposition. ■

Corollary 44.5.
(i) A CW complex X is compact if and only if it has finitely many cells.
(ii) A CW complex X is locally compact if and only if the collection $\mathcal{E}$ of cells is locally finite.

We have talked recently about two good families of spaces, CW complexes and manifolds. How are they related? A CW complex is a much more general kind of space. For instance, $S^1 \vee S^1$ has a perfectly good CW structure with a single 0-cell and two 1-cells, but it is not a manifold since the basepoint does not have a Euclidean neighborhood. On the other hand, most manifolds do admit CW decompositions.

Theorem 44.6 (Lee, 5.25)**.** *Every 1-manifold admits a (nice) CW decomposition.*

Theorem 44.7 (Lee, 5.36, 5.37)**.** *Every n-manifold admits a (nice) CW decomposition for $n = 2, 3$.*

According to p. 529 of Allen Hatcher's Algebraic Topology book, it is an open question whether or not every 4-manifold admits a CW decomposition. But n-manifolds for $n \geq 5$ do always admit a CW decomposition.

Another important problem, back purely in the realm of manifolds, is to try to list all manifolds of a given dimension.

Theorem 44.8 (Classification of 1-manifolds)**.** *Every nonempty, connected 1-manifold M is homeomorphic to S^1 if it is compact and to $\mathbb{R}$ if it is noncompact.*

For this theorem, it will be convenient to work with nice CW structures.

Definition 44.9. If X is a space with a CW structure, we say that an n-cell e_i^n is **regular** if the characteristic map $\Phi_i : D^n \longrightarrow \overline{e_i} \subset X$ is a homeomorphism onto its image. We say that a CW complex is regular if every cell is regular.

Proof. The first step is to show that every 1-manifold has a regular CW decomposition. The main idea is to cover M by a countable collection $\{U_n\}$ of regular charts (each closure $\overline{U_n}$ in M should be homeomorphic to $[0, 1]$). Then, using induction, it is possible to put a regular CW structure on $\mathcal{U}_n = \bigcup_{k=1}^n U_k$ in such a way that $\mathcal{U}_n \subseteq \mathcal{U}_{n+1}$ is the inclusion of a subcomplex. (See Lee 5.25 for more details.) Clearly, each 1-cell bounds two 0-cells, since the 1-cell is assumed to be regular. Somewhat less clear is the fact that each 0-cell is in the boundary of two 1-cells (see Lee 5.26).

We enumerate the 0-cells (aka vertices) and 1-cells (aka edges) in the following way. First, pick some 0-cell, and call it v_0. Pick an edge ending at v_0, and call this e_0. The other endpoint of e_0 we call v_1. The other edge ending at v_1 is called e_1. We can continue in this way to get $v_2, v_3, \ldots$ and $e_2, e_3, \ldots$. Now there is also another edge ending at v_0, which should be called e_{-1}. Let v_{-1} be the other endpoint. We can continue to get $v_{-2}, v_{-3}, \ldots$ and $e_{-2}, e_{-3}, \ldots$.

There are two cases to consider:

Case 1: The vertices v_i, $i \in \mathbb{Z}$ are all distinct. Then for each $n \in \mathbb{Z}$, we have an embedding $[n, n+1] \cong [-1, 1] \xrightarrow{\Phi_n} \overline{e_n^1}$. These glue together to give a continuous map $f : \mathbb{R} \longrightarrow X$. Our assumption means that f is injective when restricted to $\mathbb{Z}$. We can then see it is globally injective since its restriction to any $(n, n+1)$ is a characteristic map for a cell (thus injective) and all cells are disjoint.

Next, we show that f is open. Any open subset of $(n, n+1)$ is taken by f to an open subset of M, since the top-dimension cells are always open in a CW complex. It remains to show that f takes intervals of the form $(n - \epsilon, n + \epsilon)$ to open subsets of M. By taking ϵ small enough, we can ensure that this image is contained in (the closure of) two 1-cells. We can then see that this subset of M is open by pulling back along the characteristic maps (pulling back along these two characteristic maps will give half-open intervals in D^1).

Since M is connected, in order to show that f is surjective, it now suffices to show that $f(\mathbb{R})$ is closed. Let $x \notin f(\mathbb{R})$. If x lies in a 1-cell e, then e is a neighborhood of x disjoint from $f(\mathbb{R})$. The other possibility is that x is a 0-cell. But then x must be the endpoint of two closed 1-cells $\overline{e}$ and $\overline{e'}$. Neither e nor e' can be in $f(\mathbb{R})$ since this would imply that x also lies in $f(\mathbb{R})$. But then $e \cup \{x\} \cup e'$ is a neighborhood of x disjoint from $f(\mathbb{R})$.

We have shown that $f : \mathbb{R} \longrightarrow M$ is an open, continuous bijection. So it is a homeomorphism.

Case 2: For some $n \in \mathbb{Z}$ and $k > 0$, we have $v_n = v_{n+k}$. We may then pick n and k so that k is minimal. Then the vertices $v_n, \ldots, v_{n+k-1}$ are distinct, as are the edges $e_n, \ldots, e_{n+k-1}$. This implies that the restriction of f to $[n, n+k)$ is injective. If we consider the restriction only to the closed interval $[n, n+k]$, then we get a closed map, since the domain is compact and the target is

Hausdorff. We claim also that $f([n, n+k])$ is open in M. Indeed, if we pick any $x \in [n, n+k]$ which lies in an interval $(i, i+1)$, then the open 1-cell e_i is a neighborhood of $f(x)$ that is contained in the image of f. If we consider any interior integer $n < i < n+k$, then $e_{i-1} \cup \{v_i\} \cup e_i$ is an open neighborhood in the image of f. Finally, $e_n \cup \{n\} e_{n+k-1}$ is a neighborhood of $f(n) = f(n+k)$ in M.

Since the image $f([n, n+k])$ is both closed and open in M and M is connected, we conclude that $f([n, n+k]) = M$. Since $f(n) = f(n+k)$, we get an induced map

$$\overline{f} : [n, n+k]/\sim \,\cong S^1 \longrightarrow M$$

which is a bijection. Since S^1 is compact and M is Hausdorff, this is a homeomorphism. ■

We previously also briefly mentioned the idea of a "manifold with boundary". There is a similar result:

Theorem 44.10. *Every nonempty, connected* 1*-manifold with boundary is homeomorphic to* $[0, 1]$ *if it is compact and to* $[0, 1)$ *if it is noncompact.*

Next semester, we will similarly classify all compact 2-manifolds (the list of answers will be a little longer).

A closely related idea to CW complex is the notion of **simplicial complex**. A simplicial complex is built out of "simplices". By definition, an n**-simplex** is the convex hull of $n+1$ "affinely independent" points in $\mathbb{R}^k$, for $k \geq n+1$. This means that after translating this set so that one point moves to the origin, the resulting collection of points is linearly independent.

There is a standard n-simplex $\Delta^n \subseteq \mathbb{R}^{n+1}$ defined by

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum t_i = 1, t_i \geq 0\}.$$

In general, if σ is an n-simplex generated by $\{t_0, \dots t_n\}$, then the convex hull of any subset is called a **face** of the simplex. A (Euclidean) simplicial complex is then a subset of $\mathbb{R}^k$ that is a union of simplices such that any two overlapping simplices meet in a face of each. We also usually require the collection of simplices to be locally finite.

Since an n-simplex is homeomorphic to D^n, it can be seen that a simplicial complex is a regular CW complex. A decomposition of a manifold as a simplicial complex is known as a **triangulation** of the manifold. Just as one can ask about CW structures on manifolds, one can also ask about triangulations for manifolds.

Theorem 44.11.
(1) *Every* 1*-manifold is triangulable (indeed, we know the complete list of connected* 1*-manifolds).*
(2) *Tibor Radó proved in 1925 that every* 2*-manifold is triangulable.*
(3) *Edwin Moise proved in the 1950s that every* 3*-manifold is triangulable.*
(4) *Michael Freedman discovered the* 4*-dimensional* E_8*-manifold in 1982, which is not triangulable.*
(5) *Ciprian Manolescu showed in March 2013 that manifolds in dimension* ≥ 5 *are not triangulable.*