3. Wed, Sept. 3

In calculus, we are also used to thinking of continuity in terms of convergence of sequences. Recall that a sequence (x_n) in X **converges** to x if for every $\epsilon > 0$ there exists N such that for all $n > N$, we have $x_n \in B_\epsilon(x)$. We say that a "tail" of the sequence is contained in the ball around x.

Proposition 3.1. *The sequence (x_n) converges to x if and only if for every open set U containing x, some tail of (x_n) lies in U.*

Proof. Exercise. ∎

Proposition 3.2. *Let $f : X \longrightarrow Y$ be a function between metric spaces. The following are equivalent:*

(1) *f is continuous*
(6) *For every convergent sequence $(x_n) \to x$ in X, the sequence $(f(x_n))$ converges to $f(x)$ in Y.*

Proof. This is on HW1. ∎

This finishes our discussion of continuity.

What constructions can we make with metric spaces?

Products: Let's start with a product. That is, if (X, d_X) and (Y, d_Y) are metric spaces, is there a good notion of the product metric space? We would want to have continuous "projection" maps to each of X and Y, and we would want it to be true that to define a continuous map from some metric space Z to the product, it is enough to specify continuous maps to each of X and Y. By thinking about the case in which Z has a discrete metric, one can see that the underlying set of the product metric space would need to be the cartesian product $X \times Y$. The only question is whether or not there is a sensible metric to define.

Recall that we discussed three metrics on $\mathbb{R}^2$: the standard one, the max metric, and the taxicab metric. There, we used that $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ as an underlying set, and we combined the metrics on each copy of $\mathbb{R}$ to get a metric on $\mathbb{R}^2$. We can use the same idea here to get three different metrics on $X \times Y$, and these will all produce a metric space satisfying the right property to be a product.

For convenience, let's pick the max metric on $X \times Y$. To show that the projection $p_X : X \times Y \longrightarrow X$ is continuous, it is enough to show that each $p_X^{-1}(B_\epsilon(x))$ is open. But it is simple to show that

$$p_X^{-1}(B_\epsilon(x)) = B_\epsilon(x) \times Y$$

is open using the max metric. The same argument shows that p_Y is continuous.

Now suppose that Z is another metric space with continuous maps $f_X, f_Y : Z \rightrightarrows X, Y$. We define $f = (f_X, f_Y)$ coordinate-wise as before, and it only remains to show that it is continuous. Consider a ball $B_\epsilon(x, y) \subset X \times Y$. Under the max metric, this ball can be rewritten as

$$B_\epsilon(x, y) = B_\epsilon(x) \times B_\epsilon(y),$$

so that

$$f^{-1}(B_\epsilon(x, y)) = f^{-1}\big(B_\epsilon(x) \times B_\epsilon(y)\big) = f_X^{-1}(B_\epsilon(x)) \cap f_Y^{-1}(B_\epsilon(y)).$$

By a problem from HW1, this is open, showing that f is continuous.

Function spaces: Another important construction is that of a space of functions. That is, if X and Y are metric spaces, one can consider the set of all continuous functions $f : X \longrightarrow Y$. Is there a good way to think of this as a metric space? For example, as a set $\mathbb{R}^2$ is the same as the collection of functions $\{1, 2\} \longrightarrow \mathbb{R}$. More generally, we could consider functions $\{1, \ldots, n\} \longrightarrow Y$ or even $\mathbb{N} \longrightarrow Y$ (i.e. sequences).

Of the metrics we discussed on $\mathbb{R}^2$, the max metric generalizes most easily to give a metric on $Y^\infty = Y^{\mathbb{N}}$. We provisionally define the **sup metric** on the set of sequences in Y by

$$d_{\sup}((y_n),(z_n)) = \sup_n\{d_Y(y_n,z_n)\}.$$

Without any further restrictions, there is no reason that this supremum should always exist. If Y is a bounded metric space, or if we only consider bounded sequences, then we are OK. Another option is to arbitrarily truncate the metric.

Lemma 3.3. *Let (Y,d) be a metric space. Define the resulting bounded metric $\overline{d}$ on Y by*

$$\overline{d}(y,z) = \min\{d(y,z),1\}.$$

This is a metric, and the open sets determined by $\overline{d}$ are precisely the open sets determined by d.

We now redefine the sup metric on Y^∞ to be

$$d_{\sup}((y_n),(z_n)) = \sup_n\{\overline{d_Y}(y_n,z_n)\}.$$

Now the supremum always exists, so that we get a well-defined metric. The same definition works to give a metric on the set of continuous functions $X \longrightarrow Y$. We define the sup metric on the set $\mathcal{C}(X,Y)$ of continuous functions to be

$$d_{\sup}(f,g) = \sup_{x\in X}\{\overline{d_Y}(f(x),g(x))\}.$$

This is also called the **uniform metric**, for the following reason.

Proposition 3.4. *Let (f_n) be a sequence in $\mathcal{C}(X,Y)$. Then $(f_n)\to f$ in the uniform metric on $\mathcal{C}(X,Y)$ if and only if $(f_n)\to f$ uniformly.*

Given a function $f \in \mathcal{C}(X,Y)$ and a point $x \in X$, one can evaluate the function to get $f(x) \in Y$. In other words, we have an evaluation function

$$\text{eval} : \mathcal{C}(X,Y)\times X \longrightarrow Y.$$

Proposition 3.5. *Consider $\mathcal{C}(X,Y)\times X$ as a metric space using the max metric. Then* eval *is continuous.*

Proof. We know that to determine if a function between metric spaces is continuous, it it suffices to check that it takes convergent sequences to convergent sequences. Suppose that $(f_n,x_n)\to(f,x)$. We wish to show that

$$\text{eval}(f_n,x_n) = f_n(x_n) \to \text{eval}(f,x) = f(x).$$

Since $(f_n,x_n)\to(f,x)$, it follows that $f_n\to f$ and $x_n\to x$ (since the projections are continuous).

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Let $\varepsilon>0$. Then there exists N_1 such that if $n>N_1$ then $d_{\sup}(f_n,f)<\varepsilon/2$. By the definition of the sup metric, this implies that $d_Y(f_n(x_n),f(x_n))<\varepsilon/2$. But now f is continuous, so there exists N_2 such that if $n>N_2$ then $d_Y(f(x_n),f(x))<\varepsilon/2$. Putting these together and using the triangle inequality, if $n>N_3=\max\{N_1,N_2\}$ then $d_Y(f_n(x_n),f(x))<\varepsilon$. ■

Proposition 4.1. *Suppose $\varphi : X\times Y \longrightarrow Z$ is continuous. For each $x\in X$, define $\hat{\varphi}(x) : Y \longrightarrow Z$ by $\hat{\varphi}(x)(y) = \varphi(x,y)$. The function $\hat{\varphi}(x)$ is continuous.*

Proof. This could certainly be done directly, using convergence of sequences to test for continuity. Here is another way to do it, using the universal property of products.

Note that $\hat{\varphi}(x)$ can be written as the composition $Y \xrightarrow{i_x} X \times Y \xrightarrow{\varphi} Z$. By assumption, φ is continuous, so it suffices to know that $i_x : Y \to X \times Y$ is continuous. But recall that continuous maps into a product correspond precisely to a pair of continuous maps into each factor. The pair of maps here is the constant map $Y \longrightarrow X$ at x and the identity map $Y \longrightarrow Y$. The identity map is clearly continuous, and the constant map is continuous since if $U \subseteq X$ is open, then the preimage under the constant map is either (1) all of Y if $x \in U$ or (2) empty if $x \notin U$. So it follows that i_x is continuous. ■

We are headed to the universal property of the mapping space. Keeping the notation from above, given a continuous function

$$\varphi : X \times Y \longrightarrow Z,$$

we get a function

$$\hat{\varphi} : X \longrightarrow \mathcal{C}(Y, Z).$$

Conversely, given the function $\hat{\varphi}$, we define φ by

$$\varphi(x, y) = \hat{\varphi}(x)(y).$$

Proposition 4.2. *The function φ above is continuous if $\hat{\varphi}$ is continuous.*

Proof. On homework 2. ■

Of course, we would like this to be an if and only if, but that is only true under additional hypotheses (like Y compact, for instance.) Another way to state the if-and-only-if version of this proposition is that we get a bijection

$$\mathcal{C}(X \times Y, Z) \cong \mathcal{C}(X, \mathcal{C}(Y, Z)).$$

For those who have seen the $(\otimes, \mathrm{Hom})$ adjunction in algebra, this is completely analogous.

Quotients Another important construction that we will discuss when we move on to topological spaces is that of a quotient, or identification space. A standard example is the identification, on the unit interval $[0, 1]$, of the two endpoints. Glueing these together gives a circle S^1, and the surjective continuous map

$$e^{2\pi i x} : [0, 1] \longrightarrow S^1$$

is called the quotient map. Here the desired universal property is that if $f : [0, 1] \longrightarrow Y$ is a continuous map to another metric space such that $f(0) = f(1)$, then the map f should "factor" through the quotient. Quotients become quite complicated to express in the world of metric spaces.

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Now that we have spent some time with metric spaces, let's turn to the more general world of topological spaces.

Definition 4.3. A **topological space** is a set X with a collection of subsets $\mathcal{T}$ of X such that

(1) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$
(2) If $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$
(3) If $U_i \in \mathcal{T}$ for all i in some index set I, then $\bigcup_{i \in I} U_i \in \mathcal{T}$.

The collection $\mathcal{T}$ is called the **topology** on X, and the elements of $\mathcal{T}$ are referred to as the "open sets" in the topology.

Example 4.4.

(1) (Metric topology) Any metric space is a topological space, where $\mathcal{T}$ is the collection of metric open sets

(2) (Discrete topology) In the discrete topology, *every* subset is open. We already saw the discrete metric on any set, so in fact this is an example of a metric topology as well.

(3) (Trivial topology) In the trivial topology, $\mathcal{T} = \{\emptyset, X\}$. That is, $\emptyset$ and X are the only empty sets. This topology does not come from a metric (unless X has fewer than two points).

(4) It is simple to write down various topologies on a finite set. For example, on the set

$$X = \{1, 2\},$$

there are 4 possible topologies. In addition to the trivial and discrete topologies, there is also

$$\mathcal{T}_1 = \{\emptyset, \{1\}, X\}$$

and

$$\mathcal{T}_2 = \{\emptyset, \{2\}, X\}.$$

(5) There are many possible topologies on $X = \{1, 2, 3\}$. But not every collection of subsets will give a topology. For instance,

$$\{\emptyset, \{1, 2\}, \{1, 3\}, X\}$$

would not be a topology, since it is not closed under intersection.