At the end of class on Friday, we introduced the notion of a topology, and I asked you to think about how many possible topologies there are on a 3-element set. The answer is ... 29. The next few answers for the number of topologies on a set of size n are[1]: 355 ($n = 4$), 6942 ($n = 5$), 209527 ($n = 6$). But there is no known formula for answer in general.

When working with metric spaces, we saw that the topology was determined by the open balls. Namely, an open set was precisely a subset that could be written as a union of balls. In many topologies, there is an analogue of these basic open sets.

Definition 5.1. A **basis** for a topology on X is a collection $\mathcal{B}$ of subsets such that

(1) (Covering property) Every point of x lies in at least one basis element
(2) (Intersection property) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, then there is a third basis element B_3 such that
$$x \in B_3 \subseteq B_1 \cap B_2.$$

A basis $\mathcal{B}$ defines a topology $\mathcal{T}_\mathcal{B}$ by declaring the open sets to be the unions of (arbitrarily many) basis elements.

Proposition 5.2. *Given a basis $\mathcal{B}$, the collection $\mathcal{T}_\mathcal{B}$ is a topology.*

Proof. It is clear that open sets are closed under unions. The emptyset is a union of no basis elements, so it is open. The set X is open by the covering property: the union of all basis elements is X. Finally, we check that the intersection of two open sets is open. Let U_1 and U_2 be open. Then
$$U_1 = \bigcup_{\alpha \in A} B_\alpha, \qquad U_2 = \bigcup_{\delta \in \Delta} B_\delta.$$
We want to show that $U_1 \cap U_2$ is open. Now
$$U_1 \cap U_2 = \left(\bigcup_{\alpha \in A} B_\alpha\right) \cap \left(\bigcup_{\delta \in \Delta} B_\delta\right) = \bigcup_{\alpha \in A, \delta \in \Delta} B_\alpha \cap B_\delta.$$
It remains to show that $B_\alpha \cap B_\delta$ is open. By the intersection property of a basis, for each $x \in B_\alpha \cap B_\delta$, there is some B_x with
$$x \in B_x \subseteq B_\alpha \cap B_\delta.$$
It follows that
$$B_\alpha \cap B_\delta = \bigcup_{x \in B\alpha \cap B_\delta} B_x,$$
so we are done. ■

Example 5.3. We have already seen that metric balls form a basis for the metric topology. In the case of the discrete metric, one can take the balls with radius 1/2, which are exactly the singleton sets.

Example 5.4. For a truly new example, we take as basis on $\mathbb{R}$, the half-open intervals $[a, b)$. The resulting topology is known as the **lower limit topology** on $\mathbb{R}$.

How is this related to the usual topology on $\mathbb{R}$? Well, any open interval (a, b) can be written as a union of half-open intervals. However, the $[a, b)$ are certainly not open in the usual topology. This says that $\mathcal{T}_{\text{standard}} \subseteq \mathcal{T}_{\ell\ell}$. The lower limit topology has more open sets than the usual topology. When one topology on a set has more open sets than another, we say it is **finer**. So the lower limit

[1]These are taken from the On-Line Encyclopedia of Integer Sequences.

topology is *finer* than the usual topology on $\mathbb{R}$, and the usual topology is *coarser* than the lower limit topology.

On any set X, the discrete topology is the finest, whereas the trivial topology is the coarsest.

When a topology is generated by a basis, there is a convenient criterion for open sets.

Proposition 5.5. *(Local criterion for open sets) Let $\mathcal{T}_{\mathcal{B}}$ be a topology on X generated by a basis $\mathcal{B}$. Then a set $U \subseteq X$ is open if and only if, for each $x \in U$, there is a basis element $B_x \in \mathcal{B}$ with $x \in B_x \subseteq U$.*

Proof. ($\Rightarrow$) By definition of $\mathcal{T}_{\mathcal{B}}$, the set U is a union of basis elements, so any $x \in U$ must be contained in one of these.

($\Leftarrow$) We can write $U = \bigcup_{x \in U} B_x$. ■

This is a good time to introduce a convenient piece of terminology: given a point x of a space X, a **neighborhood** N of x in X is a subset of X containing some open set U with $x \in U \subseteq N$. Often, we will take our neighborhoods to themselves be open.

Given our discussion of continuous maps between metric spaces, it should be clear what the right notion is for maps between topological spaces.

Definition 5.6. A function $f : X \longrightarrow Y$ between topological spaces is said to be **continuous** if for every open subset $V \subseteq Y$, the preimage $f^{-1}(V)$ is open in X.

Example 5.7. Let $X = \{1, 2\}$ with topology $\mathcal{T}_X = \{\emptyset, \{1\}, X\}$ and let $Y = \{1, 2, 3\}$ with topology $\mathcal{T}_Y = \{\emptyset, \{2\}, \{3\}, \{2, 3\}, Y\}$. Which functions $X \longrightarrow Y$ are continuous?

Let's start with the open set $\{2\} \subseteq Y$. The preimage must be open, so it can either be $\emptyset$ or $\{1\}$ or X. If the preimage is X, the function is constant at 2, which is continuous.

Suppose the preimage is $\emptyset$. Then the preimage of $\{3\}$ can be either $\emptyset$ or $\{1\}$ or X. If it is $\emptyset$, we are looking at the constant function at 1, which is continuous. If $f^{-1}(3) = X$, then f is constant at 3, which is continuous. Finally, if $f^{-1}(3) = \{1\}$, then f must be the continuous function $f(1) = 3$, $f(2) = 1$.

Finally, suppose $f^{-1}(2) = \{1\}$. Then $f^{-1}(3)$ can't be $\{1\}$ or X, so the only possible continuous f has $f^{-1}(3) = \emptyset$, so that we must have $f(1) = 2$ and $f(2) = 1$.

By the way, we asserted above that constant functions are continuous. We proved this before (top of page 7) for metric spaces, but the proof given there applies verbatim to general topological spaces.

6. Wed, Sept. 10

Proposition 6.1. *Suppose $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are continuous. Then so is their composition $g \circ f : X \longrightarrow Z$.*

Proof. Let $V \subseteq Z$ be open. Then

$$\begin{aligned}(g \circ f)^{-1}(V) &= \{x \in X \mid (g \circ f)(x) \in V\} = \{x \in X \mid g(f(x)) \in V\} \\ &= \{x \in X \mid f(x) \in g^{-1}(V)\} = \{x \in X \mid x \in f^{-1}(g^{-1}(V))\} = f^{-1}(g^{-1}(V)).\end{aligned}$$

Now g is continuous, so $g^{-1}(V)$ is open in Y, and f is continuous, so $f^{-1}(g^{-1}(V))$ is open in X. ■

Another construction we can consider with continuous functions is the idea of restricting a continuous function to a subset. For instance, the natural logarithm is a nice continuous function $\ln : (0, \infty) \longrightarrow \mathbb{R}$, but we also get a nice continuous function by considering the logarithm only on $[1, \infty)$. To have this discussion here, we should think about how a subset of a space becomes a space in its own right.

Definition 6.2. Let X be a space and let $A \subseteq X$ be a subset. We define the subspace topology on A by saying that $V \subseteq A$ is open if and only if there exists some open $U \subseteq X$ with $U \cap A = V$.

Note that the open set $U \subseteq X$ is certainly not unique.

Example 6.3.
(1) Let $A = \mathbb{R} \times \{0\} \subseteq \mathbb{R}^2$. Then the subspace topology on $A \cong \mathbb{R}$ is the usual topology on $\mathbb{R}$. Indeed, consider the usual basis for $\mathbb{R}^2$ consisting of open disks. Intersecting these with A gives open intervals. In general, intersecting a basis for X with a subset A gives a basis for A, and here we clearly get the usual basis for the standard topology. The same would be true if we started with max-metric basis (consisting of open rectangles).
(2) Let $A = (0, 1) \subseteq X = \mathbb{R}$. We claim that $V \subseteq A$ is open in the subset topology if and only if V is open as a subset of $\mathbb{R}$. Indeed, suppose that V is open in A. Then $V = U \cap (0, 1)$ for some open U in $\mathbb{R}$. But now both U and $(0, 1)$ are open in $\mathbb{R}$, so it follows that their intersection is as well. The converse is clear.

    Note that this statement fails for the previous example. $(0, 1) \times \{0\}$ is open in A there but not open in $\mathbb{R}^2$.
(3) Let $A = (0, 1]$. Then, in the subspace topology on A, every interval $(a, 1]$, with $a < 1$ is an open set. A basis for this topology on A consists in the (a, b) with $0 \leq a < b < 1$ and the $(a, 1]$ with $0 \leq a < 1$.
(4) Let $A = (0, 1) \cup \{2\}$. Then the singleton $\{2\}$ is an open subset of A! A basis consists of the (a, b) with $0 \leq a < b \leq 1$ and the singleton $\{2\}$.

Given a subset $A \subseteq X$, there is always the inclusion function $\iota_A : A \longrightarrow X$ defined by $\iota_A(a) = a$.

Proposition 6.4. *Given a subset $A \subseteq X$ of a topological space, the inclusion ι_A is continuous. Moreover, the subspace topology on A is the coarsest topology which makes this true.*

Proof. Suppose that $U \subseteq X$ is open. Then $\iota_A^{-1}(U) = U \cap A$ is open in A by the definition of the subspace topology.

To see that this is the coarsest such topology, suppose that $\mathcal{T}'$ is a topology which makes the inclusion $\iota_A : A \longrightarrow X$. We wish to show that $\mathcal{T}'$ is finer than the subspace topology, meaning that $\mathcal{T}_A \subseteq \mathcal{T}'$, where $\mathcal{T}_A$ is the subspace topology. So let V be open in $\mathcal{T}_A$. This means there exists $U \subseteq X$ open such that $V = U \cap A = \iota_A^{-1}(U)$. Since ι_A is continuous according to $\mathcal{T}'$, it follows that V is open in $\mathcal{T}'$. ■

Getting back to our motivational question, suppose that $f : X \longrightarrow Y$ is continuous and let $A \subseteq X$ be a subset. We define the restriction of f to A, denoted $f_{|A}$, by

$$f_{|A} : A \longrightarrow Y, \qquad f_{|A}(a) = f(a).$$

Proposition 6.5. *Let $f : X \longrightarrow Y$ be continuous and suppose that $A \subseteq X$ is a subset. Then the restriction $f_{|A} : A \longrightarrow Y$ is continuous.*

Proof. This is just the composition $f_{|A} = f \circ \iota_A$. ■

So far, we only discussed the notion of open set, but there is also the complementary notion of closed set.

Definition 6.6. Let X be a space. We say a subset $W \subseteq X$ is **closed** if the complement $X \setminus W$ is open.

Note that, despite what the name may suggest, closed does *not* mean "not open". For instance, the empty set is always both open (required for any topology) and closed (because the complement, X must be open). Similarly, there are many examples of sets that are neither open nor closed (for example, the interval $[0, 1)$ in the usual topology on $\mathbb{R}$).

Proposition 6.7. *Let X be a space.*

(1) *$\emptyset$ and X are both closed in X*
(2) *If W_1, W_2 are closed, then $W_1 \cup W_2$ is also closed*
(3) *If W_i are closed for all i in some index set I, then $\bigcap_{i \in I} W_i$ is also closed.*

Proof. We prove (2). The point is that

$$X \setminus (W_1 \cup W_2) = (X \setminus W_1) \cap (X \setminus W_2).$$

This equality is known as one of the DeMorgan Laws ■

7. Fri, Sept. 12

Last time, we defined the notion of a closed set.

Example 7.1. Consider $\mathbb{R}_{\ell\ell}$, the real line equipped with the lower-limit topology. (Example 5.4). There, a half-open interval $[a, b)$ was declared to be open. It then follows that intervals of the form $(-\infty, b)$ and $[a, \infty)$ are open. But this then implies that $[a, b)$ is *closed* since its complement is the open set $(-\infty, a) \cup [b, \infty)$.

Not only does a topology give rise to a collection of closed sets satisfying the above properties, but one can also define a topology by specifying a list of closed sets satisfying the above properties.

Similarly, we can use closed sets to determine continuity.

Proposition 7.2. *Let $f : X \longrightarrow Y$. Then f is continuous if and only if the preimage of every closed set in Y is closed in X.*

Example 7.3. The "distance from the origin function" $d : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is continuous (follows from HW 2). Since $\{1\} \subseteq \mathbb{R}$ is closed, it follows that the sphere $S^2 = d^{-1}(1)$ is closed in $\mathbb{R}^3$. More generally, S^{n-1} is closed in $\mathbb{R}^n$.

Example 7.4. Let X be any metric space, let $x \in X$, and let $r > 0$. Then the ball

$$B_{\leq r}(x) = \{y \in X \mid d(x, y) \leq r\}$$

is closed in X.

Remark 7.5. Note that some authors use the notation $\overline{B_r(x)}$ for the closed ball. This is a bad choice of notation, since it suggests that the closure of the open ball is the closed ball. But this is not always true! For instance, consider a set (with more than one point) equipped with the discrete metric. Then $B_1(x) = \{x\}$ is already closed, so it is its own closure. On the other hand, $B_{\leq 1}(x) = X$.

Consider the half-open interval $[a, b)$. It is neither open nor closed, in the usual topology. Nevertheless, there is a closely associated closed set, $[a, b]$. Similarly, there is a closely associated open set, (a, b). Notice the containments

$$(a, b) \subseteq [a, b) \subseteq [a, b].$$

It turns out that this picture generalizes.

Let's start with the closed set. In the example above, $[a, b]$ is the *smallest closed set containing* $[a, b)$. Why should we expect such a smallest closed set to exist in general? Recall that if we intersect arbitrarily many closed sets, we are left with a closed set.

Definition 7.6. Let $A \subseteq X$ be a subset of a topological space. We define the **closure of A in X** to be

$$\overline{A} = \bigcap_{A \subset B \text{ closed}} B.$$

Dually, we have $(a,b) \subset [a,b)$, and (a,b) is the largest open set contained in $[a,b)$.

Definition 7.7. Let $A \subseteq X$ be a subset of a topological space. We define the **interior of A in X** to be

$$\operatorname{Int}(A) = \bigcup_{A \supset U \text{ open}} U.$$

The difference of these two constructions is called the **boundary of A in X**, defined as

$$\partial A = \overline{A} \setminus \operatorname{Int}(A).$$

Example 7.8.
(1) From what we have already said, it follows that $\partial[a,b) = \{a,b\}$.
(2) Let $A = \{1/n \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$. Then A is not open, since no neighborhood of any $1/n$ is contained in A. This also shows that $\operatorname{Int}(A) = \emptyset$. But neither is A closed, because no neighborhood of 0 is contained in the complement of A. This implies that $0 \in \overline{A}$, and it turns out that $\overline{A} = A \cup \{0\}$. Thus $\partial A = \overline{A} = A \cup \{0\}$.
(3) Let $\mathbb{Q} \subseteq \mathbb{R}$. Similarly to the example above, $\operatorname{Int}(\mathbb{Q}) = \emptyset$. But since $\mathbb{R} \setminus \mathbb{Q}$ does not entirely contain any open intervals, it follows that $\overline{\mathbb{Q}} = \mathbb{R}$. (A subset $A \subseteq X$ is said to be **dense** in X if $\overline{A} = X$.) Thus $\partial \mathbb{Q} = \mathbb{R} \setminus \emptyset = \mathbb{R}$.
(4) Let's turn again to $\mathbb{R}_{\ell\ell}$. We saw that $[0,1)$ was already closed. What about $(0,1]$? Since $[0,1]$ is closed in the usual topology, this must be closed in $\mathbb{R}_{\ell\ell}$ as well. (Recall that the topology on $\mathbb{R}_{\ell\ell}$ is finer than the standard one). It follows that $(0,1]$ is either already closed, or its closure is $[0,1]$. We can ask, dually, whether the complement is open. But $(-\infty,0] \cup (1,\infty)$ is not open since it does not contain any neighborhoods of 0. It follows that $\overline{(0,1]} = [0,1]$ in $\mathbb{R}_{\ell\ell}$.

There is a convenient characterization of the closure, which we were implicitly using above.

Proposition 7.9 (Neighborhood criterion)**.** *Let $A \subseteq X$. Then $x \in \overline{A}$ if and only if every neighborhood of x meets A.*

Proof. ($\Rightarrow$) Suppose $x \in \overline{A}$. Then $x \in B$ for all closed sets B containing A. Let N be a neighborhood of x. Without loss of generality, we may suppose N is open. Now $X \setminus N$ is closed but $x \notin X \setminus N$, so this set cannot contain A. This means precisely that $N \cap A \neq \emptyset$.

($\Leftarrow$) Suppose every neighborhood of x meets A. Let $A \subset B$, where B is closed in X. Now $U = X \setminus B$ is an open set not meeting A, so it cannot be a neighborhood of x. This must mean that $x \notin X \setminus B$, or in other words $x \in B$. Since B was arbitrary, it follows that x lies in every such B. ■

In our earlier discussion of metric spaces, we considered convergence of sequences and how this characterized continuity. This is one statement from the theory of metric spaces that will not carry over to the generality of topological spaces.

Definition 7.10. We say that a sequence x_n in X converges to x in X if every neighborhood of x contains a tail of (x_n).

The following result follows immediately from the previous characterization of the closure.

Proposition 7.11. *Let (a_n) be a sequence in $A \subseteq X$ and suppose that $a_n \to x \in X$. Then $x \in \overline{A}$.*

Proof. We use the neighborhood criterion. Thus let U be a neighborhood of x. Since $a_n \to x$, a tail of (a_n) lies in U. It follows that $U \cap A \neq \emptyset$, so that $x \in \overline{A}$. ■

However, we will see next time that the converse fails in general.