8. Mon, Sept. 15

Last time, we saw that if (a_n) is a sequence in $A \subseteq X$ and $a_n \to x$, then $x \in \overline{A}$. But the converse is not true in a general topological space. (The fact that these are equivalent in a metric space is known as the **sequence lemma**.) To see this, consider $\mathbb{R}$ equipped with the *cocountable* topology. Recall that this means that the nonempty open subsets are the cocountable ones.

Lemma 8.1. *Suppose that $x_n \to x$ in the cocountable topology on $\mathbb{R}$. Then (x_n) is eventually constant.*

Proof. Write B for the set

$$B = \{x_n \mid x_n \neq x\}.$$

Certainly B is countable, so it is closed. By construction, $x \notin B$, so $N = X \setminus B$ is an open neighborhood of x. But $x_n \to x$, so a tail of this sequence must lie in N. Since $\{x_n\} \cap N = \{x\}$, this means that a tail of this sequence is constant, in other words, the sequence is eventually constant. ∎

Now consider $A = \mathbb{R} \setminus \{0\} \subseteq \mathbb{R}$ in the cocountable topology. A is not closed since the only closed proper subsets are the countable ones. It follows that A must be dense in $\mathbb{R}$. However, no sequence in A can converge to 0 since a convergent sequence must be eventually constant.

Similarly, we cannot use convergence of sequences to test for continuity in general topological spaces. For instance, consider the identity map

$$\mathrm{id} : \mathbb{R}_{\text{cocountable}} \longrightarrow \mathbb{R}_{\text{standard}},$$

where the domain is given the cocountable topology and the codomain is given the usual topology. This is not continuous, since the interval $(0, 1)$ is open in $\mathbb{R}_{\text{standard}}$ but not in $\mathbb{R}_{\text{cocountable}}$. On the other hand, the identity function takes convergent sequences in $\mathbb{R}_{\text{cocountable}}$, which are necessarily eventually constant, to convergent sequences in $\mathbb{R}_{\text{standard}}$. This follows from the following result, which you proved on HW1.

Proposition 8.2. *Let $f : X \longrightarrow Y$ be continuous. If $x_n \to x$ in X then $f(x_n) \to f(x)$ in Y.*

Proof. Suppose $x_n \to x$. Let V be an open neighborhood of $f(x)$. Then, since f is continuous, $f^{-1}(V)$ is an open neighborhood of x. Thus some tail of (x_n) lies in $f^{-1}(V)$, which means that the corresponding tail of $(f(x_n))$ lies in U. ∎

However, all hope is not lost, since the following is true.

Proposition 8.3. *Let $f : X \longrightarrow Y$. Then f is continuous if and only if*

$$f(\overline{A}) \subseteq \overline{f(A)}$$

for every subset $A \subseteq X$.

Proof. ($\Rightarrow$) Assume f is continuous. Since $\overline{f(A)}$ is the intersection of *all* closed sets containing $f(A)$, it suffices to show that if B is such a closed set, then $f(\overline{A}) \subseteq B$. Well, $f(A) \subseteq B$, so

$$A = f^{-1}(f(A)) \subseteq f^{-1}(B).$$

Now f is continuous and B is closed, so by definition of the closure, we must have

$$\overline{A} \subseteq f^{-1}(B).$$

Applying f then gives $f(\overline{A}) \subseteq f(f^{-1}(B)) \subseteq B$.

($\Leftarrow$) Suppose that the above subset inclusion holds, and let $B \subseteq Y$ be closed. Let $A = f^{-1}(B)$. We wish to show that A is closed, i.e. that $\overline{A} = A$. Since $f(f^{-1}(B)) \subseteq B$, we know that

$$f(\overline{A}) \subseteq \overline{f(A)} \subseteq \overline{B} = B.$$

Applying f^{-1} gives

$$\overline{A} = f^{-1}(f(\overline{A})) \subseteq f^{-1}(B) = A.$$

It follows that A is closed. ■

Ok, so we have learned that points in $\overline{A}$ are good enough to determine continuity of functions, but these points are not necessarily limits of sequences in A. It turns out that there is an alternative characterization of these points.

Definition 8.4. Let X be a space and $A \subseteq X$. A point $x \in X$ is said to be an **accumulation point** (or **cluster point** or **limit point**) of A if

every neighborhood of x contains a point of A other than x itself.

We sometimes write A' for the set of accumulation points of A.

Example 8.5. (1) Let $A = (0,1) \subseteq \mathbb{R}$. Then $A' = [0,1]$.
(2) Let $A = \{0,1\} \subseteq \mathbb{R}$. Then $A' = \emptyset$.
(3) Let $A = [0,1) \cup \{2\}$. Then $A' = [0,1]$.
(4) Let $A = \{1/n\} \subseteq \mathbb{R}$. Then $A' = \{0\}$.

The following result follows immediately from our neighborhood characterization of the closure of a set.

Proposition 8.6. *A point x is an acc. point of A if and only if $x \in \overline{A \setminus \{x\}}$.*

Certainly $A \setminus \{x\} \subseteq A$, and the closure operation preserves containment, so it follows that $A' \subseteq \overline{A}$. From the previous examples, we see that this need not be an equality. We also have $A \subseteq \overline{A}$, and it follows that

$$A \cup A' \subseteq \overline{A}.$$

Proposition 8.7. *For any subset $A \subseteq X$, we have*

$$A \cup A' = \overline{A}.$$

Proof. It remains to show that every point in the closure is either in A or in A'. Let $x \in \overline{A}$, but suppose that $x \notin A$. Then, by the neighborhood criterion, we have that for every neighborhood N of x, $N \cap A \neq \emptyset$. But since $x \notin A$, it follows that $N \cap (A \setminus \{x\}) \neq \emptyset$. In other words, $x \in A'$. ■

9. Wed, Sept. 17

Note that, although the motivation came from looking at sequences, there is no direct relation between accumulation points of A and limits of sequences in A.

We already saw an example of a point in the closure which is not the limit of a sequence. On the other hand, we can ask

Question 9.1. *If (a_n) is a sequence in A and $a_n \to x$, is $x \in A'$?*

Answer. No. Take $A = \{x\}$ and $a_n = x$. But, if we require that $x \notin A$, then the answer is yes.

As the example $X = \mathbb{R}^n$ suggests, sequences and closed sets are much better behaved for metric spaces.

Proposition 9.2 (The sequence lemma)**.** *Let $A \subseteq X$ and suppose that X is a metric space. Then $x \in \overline{A}$ if and only if x is the limit of a sequence in A.*

Proof. Suppose $a_n \to x$. Then either $x \in A'$, in which case we are done. Otherwise, there must be a neighborhood N of x such that $N \cap (A \setminus \{x\}) = \emptyset$. But $N \cap \{a_n\} \neq \emptyset$, so it must be that $x = a_n$ for some n, in other words $x \in A$.

On the other hand, suppose $x \in \overline{A}$. If $x \in A$, we can just take a constant sequence, so suppose not. For each n, $B_{1/n}(x)$ is a neighborhood of x, and $x \in \overline{A}$, so $B_{1/n}(x) \cap A \neq \emptyset$. Let $a_n \in B_{1/n}(x) \cap A$. Then the sequence $a_n \to x$, and $a_n \in A$ by construction. ■

The last few lectures, we have seen that closed sets are not as easily understood in general as they are in the case of metric spaces. Although we will not want to restrict ourselves to metric spaces, it will nevertheless be helpful to have some good characterizations of the "reasonable" spaces. We mention here a few of these properties.

Definition 9.3. A space X is said to be Hausdorff (also called T_2) if, given any two points x and y in X, there are disjoint open sets U and V with $x \in U$ and $y \in V$.

This is a somewhat mild "separation property" that is held by many spaces in practice and that also has a number of nice consequences.

The Hausdorff property forces sequences to behave well, in the following sense.

Proposition 9.4. *In a Hausdorff space, a sequence cannot converge simultaneously to more than one point.*

Proof. Suppose $x_n \to x$ and $x_n \to y$. Every neighborhood of x contains a tail of x_n, as does any neighborhood of y. It follows that no neighborhood of x is disjoint from any neighborhood of y. Since X is Hausdorff, this forces $x = y$. ■

Proposition 9.5. *Every metric space is Hausdorff.*

Proof. If $x \neq y$, let $d = d(x, y) > 0$. Then the balls of radius $d/2$ centered at x and y are the needed disjoint neighborhoods. ■

However, of the (many, many) topologies on a finite set, the only one that is Hausdorff is the discrete topology. Indeed, if points are closed, then every subset is closed, as it is a finite union of points.

Another property of metric spaces that we used recently was the existence of the balls of radius $1/n$.

Definition 9.6. A space X is **first-countable** if, for each $x \in X$, there is a countable collection $\{U_n\}$ of neighborhoods of x such that any other neighborhood contains at least one of the U_n.

This was the key property used in proving that, in a metric space, an accumulation point of $A \subseteq X$ is the limit of an A-sequence.

Example 9.7. The space $X = \mathbb{R}_{\text{cocountable}}$ is not first countable. To see this, let $x \in X$ and suppose that $\{U_n\}$ is a collection of neighborhoods of x. By definition, each U_n is open and misses only countably many real numbers. Write $C_n = \mathbb{R} \setminus U_n$. Then $C = \bigcup_n C_n$ is also countable, and it follows that $U = X \setminus C$ is a neighborhood of x. But U does not contain any U_n because if $U_n \subseteq U$, this would mean that $C_n \supseteq C$. Instead, we see that $C_n \subseteq C$, so that $U \subseteq U_n$ for all n. The above argument is not quite careful enough, since all of the above inclusions could be equalities. To fix it, simply note that the countable set $\bigcup_n C_n$ cannot be all of $\mathbb{R}$, since it is countable. Let C' be the union C, but with one extra element of $\mathbb{R}$ added in. Then C' is still countable, and each C_n is strictly contained in C.

Similarly, we have

Proposition 9.8. *Let $f : X \longrightarrow Y$ be a function, where X is first-countable. Then f is continuous if and only if f takes convergent sequences in X to convergent sequences in Y.*

We will return to first-countable (and second-countable) spaces later in the course.

10. Fri, Sept. 19

Last time, there was a discussion of Hausdorff spaces. Here is one more nice consequence of this property.

Proposition 10.1. *If X is Hausdorff, then points are closed in X. (A space is called T_1 if points are closed.)*

Proof. The neighborhood crtierion for the complement $X \setminus \{x\}$ is easy to verify. ∎

In Calculus, you saw functions defined piecewise, and one-sided limits were typically employed to establish continuity. There is an analogue of this type of construction for spaces.

Lemma 10.2 (Glueing/Pasting Lemma)**.** *Let $X = A \cup B$, where either (1) both A and B are open in X or (2) both A and B are closed in X. Then a function $f : X \longrightarrow Y$ is continuous if and only if the restrictions $f_{|A}$ and $f_{|B}$ are both continuous.*

Proof. ($\Rightarrow$) We already proved this in Proposition 6.5.

($\Leftarrow$) We give the proof assuming they are both open. Let $V \subseteq Y$ be open. We wish to show that $f^{-1}(V) \subseteq X$ is open. Let's restrict to A. We have $f^{-1}(V) \cap A = f_{|A}^{-1}(V)$. Since $f_{|A}$ is continuous, it follows that $f_{|A}^{-1}(V)$ is open (in A). Since A is open in X, it follows that $f_{|A}^{-1}(V)$ is also open in X. The same argument shows that $f^{-1}(V) \cap B$ is open in X. It follows that their unoin, which is $f^{-1}(V)$, is open in X. ∎

Example 10.3. For example, we can use this to paste together the continuous absolute value function $f(x) = |x|$, as a function $\mathbb{R} \longrightarrow \mathbb{R}$. We get this by pasting the continuous functions $\iota : [0, \infty) \longrightarrow \mathbb{R}$, $x \mapsto x$, and $(-\infty, 0] \cong [0, \infty) \longrightarrow \mathbb{R}$, $x \mapsto -x$.

Example 10.4. Let's look at an example of a discontinuous function, for example

$$f(x) = \begin{cases} 1 & x \neq 1 \\ 2 & x = 1. \end{cases}$$

We can get this by pasting together two constant functions, but the domains are $\mathbb{R} \setminus \{1\}$ and $\{1\}$, one of which is open but not closed, and the other of which is closed but not open.

Example 10.5. Let $X = [0, 1] \cup [2, 3]$, given the subspace topology from $\mathbb{R}$. Note that in this case each of the subsets $A = [0, 1]$ and $B = [2, 3]$ is **both** open and closed, so we can specify a continuous function on X by giving a pair of continuous functions, one on A and the other on B.

Finally, we start to look at the idea of sameness. Two sets are thought of as the same if there is a bijection between them. A bijection is simply an invertible function. More generally, we have the following idea.

Definition 10.6. A "morphism" $f : X \longrightarrow Y$ is said to be an **isomorphism** if there is a $g : Y \longrightarrow X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

Again, an isomorphism between sets is simply a bijection. In topology, this is called a **homeomorphism**. In other words, a homeomorphism is a continuous function with a continuous inverse. Since such a map is invertible, clearly it must be one-to-one and onto, but it is **not** true that every continuous bijection is a homeomorphism. Before we look at some examples, let's look at some non-examples.

Example 10.7. (1) Any time a set is equipped with two topologies, one of which is a refinement of the other, the identity map is a continuous bijection (in one direction) that is not a homeomorphism. For instance, we have the following such examples

$$\mathrm{id} : \mathbb{R} \longrightarrow \mathbb{R}_{\text{cofinite}}, \qquad \mathrm{id} : \mathbb{R}_{\text{cocountable}} \longrightarrow \mathbb{R}_{\text{cofinite}} \qquad \mathrm{id} : \mathbb{R}_{\text{discrete}} \longrightarrow \mathbb{R}$$

(2) Consider the exponential map $\exp : [0,1) \longrightarrow S^1$ given by $\exp(x) = e^{2\pi i x}$. This is a continuous bijection, but it is not a homeomorphism. Since homeomorphisms have continuous inverses, they must take open sets to open sets and closed sets to closed sets. But we see that exp does not take the open set $U = [0, 1/2)$ to an open set in S^1. The point $\exp(0) = (1, 0)$ has no neighborhood that is contained in $\exp(U)$.